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IMPACT ON MULTILAYERED COMPOSITE PLATES

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B.S. Kim and F. C. Moon

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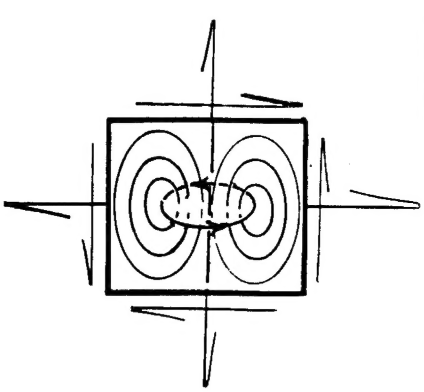

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16. Abstract

Stress wave propagation in a multilayer composite plate due to impact has been examined by means of the anisotropic elasticity theory. The plate is modelled as a number of identical anisotropic layers and the approximate plate theory of Mindlin is then applied each layer to obtain a set of difference-differential equations of motion. Dispersion relations for harmonic waves and correction factors are found. The governing equations are reduced to difference equations via integral transforms. With given impact boundary conditions these equations are solved for an arbitrary number of layers in the plate and the transient propagation of waves is calculated by means of a Fast Fourier Transform algorithm.

The multilayered plate problem is extended to examine the effect of damping layers present between two elastic layers. A reduction of the interlarminar normal stress is significant when the thickness of the damping layer is increased but it seems that the effect is mostly due to the softness of the damping layer. Finally the problem of a composite plate with a crack on the interlarminar boundary has been formulated.

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SYMBOLS USED

Δ: Plate thickness (Nondimensional length unit)

b: A half of the layer thickness

N: Layer number in the plate

ρ: Density

t_o: Impact time

$P_o(x_1,t)$: Impact stress

* $x_1(\eta)$, $x_2(\xi)$: Coordinate variables

* $t(\tau)$: Time variable

T_o: Nondimensional time unit = $a/\sqrt{c_{66}/\rho}$

* σ_{ij}, $\sigma(\Sigma)$, $\tau(T)$, σ_{11}: Stress tensor and its components

* u_i, $u(U)$, $v(V)$: Displacement vector and its components

* $c_{ijk\ell}$, $c_{ij}(C_{ij})$: Elastic Moduli $(\hat{c}_{11} = c_{11} - \frac{c_{12}^2}{c_{22}})$

λ, μ: Lame's constants

ε_{ij}, ε_i: Infinitesimal strain tensor

$\hat{A}$: Laplace transform (in τ) and Fourier transform (in η) of A

$P_n(\xi)$: Legendre polynomial of ξ

* $k(\kappa)$: Wave number

* $\omega(\bar{\omega})$: Frequency

θ, α, β: Phase shifts (wave number through the thickness)

C_D, C_S: Dilatational and shear wave speed

$G^*(\omega) = G'(\omega) + iG''(\omega)$: Complex modulus of elastomer

D: Thickness of viscoelastic layer

h: A half of the crack length

* Quantities in ( ) are nondimensional quantities.

Preface and Summary

This report is the last of a series on the response of composite plates to impact forces. The motivation for these studies has been an attempt to understand the damage to aircraft jet engine fan blades by foreign object impact such as ice balls, stones, and birds. In addition, the National Aeronautics and Space Administration, sponsors of this research, have sought to develop computer codes from these analyses which will aid the fan blade designer in locating potential failure modes and positions and thus assist in optimizing fan blade fabrication to create greater impact tolerance.

The basic approach of the principal investigator in these studies has been to use wave propagation techniques to model the early response of composite plates to impact type forces. In using the wave method, the plate can be simplified in the analyses by neglecting reflections from edge boundaries far from the impact point. Thus, while the overall geometry of the plate is no longer included in the analysis, more sophisticated mathematical models near the point of impact have been used.

The basic model for the composite plate studies has been the anisotropic plate theory as extended by Mindlin [1] to account for wave phenomena. The plate equations were used as an approximation of the exact theory of elasticity because they lead to simpler computational schemes for evaluating average stresses and displacements in the plate.

Fourier and Laplace transform techniques have been used throughout these studies and inversion of the transforms has been accomplished with a fast Fourier transform algorithm. This algorithm is an effective computational tool but requires careful scaling of the impact problem in both space and time

variables. When it is properly used it can lead to calculations of thousands of stress values in a fraction of the time of conventional finite element schemes.

In summary, the use of plate models for the fan blade impact has avoided the analytical complexities of the exact theory of elasticity as well as the computational difficulties of finite element methods.

In earlier reports both central and edge impact of an anisotropic plate were studied, [2-4]. In those reports only wave propagation in the plane of the plate was investigated. In another report [5] a multilayer plate model was developed in order to study impact induced wave propagation in both the thickness and inplane directions. In this final report further results are presented from the multilayer model. The composite plate has been modeled with as many as eight separate layers. Each layer may itself have several plys, so that effective anisotropic constants must be used for each layer in the analysis. The mathematical model exhibits wave propagation in both the thickness and inplane directions. Impact generated waves are shown to lead to interlaminar shear stresses and interlaminar tensile stresses during and after impact.

This report also presents an analysis of an impact damping mechanism. This consists of thin damping layer introduced between composite layers in the mathematical model. The resulting response due to impact shows that considerable reduction of stress can be achieved. However it appears that this stress reduction is linked to the lower elastic moduli of the damping sublayers and not the viscous nature of the sublayer.

Fianlly an attempt was made to analyze the impact of a plate with a crack. While the problem has been formulated, no progress was made on obtaining numerical answers to the crack problem.

I. INTRODUCTION

The present research is a continuation of our previous work on the stress wave propagation in a laminated composite [2-5]. It has been motivated by the problem of the impact on jet engine fan blades caused by ingestions of foreign materials, such as birds and hailstones. The successful application of fiber-reinforced composite materials depends on the ability of these materials to withstand forces due to such impact.

The simplest approach to examine the dynamic behavior of a composite plate is based upon the work of White and Angona [6]. In their work, referred to as the effective modulus theory, the response of the composite plate to waves propagating normal to the laminate is predicted by a single constant wave speed, regardless of the internal structure of composites. Even though this theory is satisfactory for many problems, it fails in the case of some problems when the wave lengths become short. To overcome this limitation, Sun and et al. proposed a model which includes the effects of internal structure, such as the layer thickness [7]. In their work, referred to so the effective stiffness theory, displacements of both the reinforcing layer and the matrix layer are expressed as linear expansions about the midplanes of the layers and approximate equations of motion are derived for both layers. Then these approximate equations are required to satisfy the continuity conditions of displacement and stress components on every interface. Using this model the propagation of harmonic waves has been examined.

More recently a number of researchers have presented models for multilayer plates either by the discrete-continuum theory or the continuum mixture theory [8-14]. Many, however examined only the frequency-wave number dispersion relationship and stopped short of the transient

impact problem except for a few experimental or numerical works using the finite element method which sometimes show a considerable discrepancy from the experimental results.

In this report we present a new attempt to mathematically model the multilayer plate and develop a method by which we can examine the transient propagation of an impact wave in the plate, not only along the longitudinal direction but also through the thickness direction of the plate as well, using an inexpensive Fast Fourier Transform algorithm [3,15].

The composite plate under consideration for the first part of the present report is imagined to comprise N identical elastic layers. And each layer is made of a number of unidirectional plys lying alternately at a layup angle $\pm\phi$ from the symmetry axis,as shown in Fig. 1. Then the elastic properties of the plate depend on the layup angle ϕ. A key assumption for the first step of the work is that all the layers are identical. While restricting the application, this assumption allows us to formulate the problem using difference-differential equations due to a rather simple periodic structure of the plate. This technique for periodic structures has been widely used in the study of electrical transmission lines [16] and in the vibration of multistory buildings [17]. By means of an approximate plate theory of Mindlin [18], a set of approximate equations of motion is developed for a typical layer using the interlaminar stresses as explicit variables. The relative motion of a layer to the adjacent layers is related by phase shifts which represent the solution of the difference parts of equations. In this way the number of the layers can be increased without increasing the size of matrix in the numerical process of invert to satisfy the boundary conditions.

It is also well understood that a thin viscoelastic layer present between elastic layers can reduce the elastic impact energy significantly by dissipating the strain energy into heat [19,20]. In our previous work [5] an elastomer layer is presented between a composite half space and a protection strip on the edge on which the impact is applied. Numerical results of the work showed an appreciable reduction in the normal stress. As an extension of this research and the first part of this report we now examine the wave propagation in a composite plate made of two elastic layers and an elastomer layer. Generalization of this problem is straightforward by assuming that our new periodic composite layer is now made of an elastic sublayer and a viscoelastic sublayer lying alternately. We can now develop the approximate theory which includes both sub-layers. For the second part of the present research we will examine the simplest case of this kind, i.e., an impact on a composite plate consisting of two elastic layers and an elastomer layer between them.

Another possible extension of the multilayer composite plate which can be found in frequent practice is discussed in the last part of this report. In this chapter a crack is introduced on the interface between two elastic layers which represent the final step before a failure occurs in the composite either by spalling or by excessive shear stress. Such crack problems constitute the main part of the study of fracture mechanics. A serious mathematical difficulty arises even in the static problems because of the mixed boundary conditions along the crack direction. The difficulty becomes more serious in the case of dynamic problems due to the diffraction of waves at the crack tip [21-24]. By employing the approximate equations of motion developed in the first part, the transient wave problem has been formulated and dual integral

equations are obtained after application of the mixed boundary conditions. But the resulting dual integral equations are not easy to solve and are under investigation at this time.

In the results presented in this report only a line impact has been examined. This has simplified the calculations and saved computer time in testing the model. The technique, however, can be extended to the two-dimensional or central impact problem. Since the impact speed is very high (~450 m/sec) and the impact time is short (< 100 μsec), the impact can be in the range of the elastic-plastic impact or even in the range of the hydraulic impact. But the initial transmission of impact energy is propagated by elastic waves, as if in an unbounded plate, and it is useful to investigate the problem by means of the linear theory of anisotropic elasticity in an infinite composite plate.

II. IMPACT ON MULTILAYER ELASTIC PLATE

1. Formulation

Basic Theory of Linear Anisotropic Elasticity

Cauchy's equations of motion in cartesian tensor form without body forces are given by

$$
\begin{aligned}
\sigma_{ij,i} &= \rho \ddot{u}_j \\
\sigma_{ij} &= \sigma_{ji}
\end{aligned}
\qquad \text{(II-1)}
$$

where the repeated index implies summation on that index. A comma represents a partial differentiation with respect to the index following the comma and a superposed dot denotes a time derivative.
tensor is related to the infinitesimal strain tensor ε_{ij} by

$$\sigma_{ij} = c_{ijk\ell}\,\varepsilon_{k\ell} \quad \text{or} \quad \sigma_i = c_{ij}\,\varepsilon_j \quad . \qquad \text{(II-2)}$$

The condensed elastic moduli c_{ij} has the following form for orthotropic materials

$$
c_{ij} = \begin{bmatrix}
c_{11}, & c_{12}, & c_{13}, & 0, & 0, & 0 \\
c_{12}, & c_{22}, & c_{23}, & 0, & 0, & 0 \\
c_{13}, & c_{23}, & c_{33}, & 0, & 0, & 0 \\
0, & 0, & 0, & c_{44}, & 0, & 0 \\
0, & 0, & 0, & 0, & c_{55}, & 0 \\
0, & 0, & 0, & 0, & 0, & c_{66}
\end{bmatrix}
$$

Analysis of a Layer

For a layer shown in Fig. 1 we employ the approximate plate theory of Mindlin [18] and the displacement field $\underset{\sim}{u}$ is expanded in terms of the Legendre polynomials as

$$\underset{\sim}{u}(x_1,x_2,x_3,t) = \sum_{n=0}^{\infty} \underset{\sim}{u}^{(n)}(x_1,x_3,t) \cdot P_n(\xi) \tag{II-3}$$

where ξ is the local coordinate along the thickness direction, normalized by b, a half layer thick.

Instead of solving Eq. (II-1) directly we solve a new approximate equation of motion which is obtained through a variational process by integration of Eq. (II-1) over the thickness ξ (see [1], [23]). The result is

$$b \cdot \sigma^{(n)}_{\alpha j\alpha} + [P_n(\xi)\cdot\sigma_{2j}]^{1}_{\xi=-1} - \overset{*}{\sigma}{}^{(n)}_{2j} = \frac{2\rho b}{2n+1}\ddot{u}^{(n)}_{j} \quad : \quad \begin{matrix} j = 1,2,3 \\ \alpha = 1,3 \end{matrix} \tag{II-4}$$

where

$$\sigma^{(n)}_{\alpha j} = \int_{-1}^{1} P_n(\xi)\cdot\sigma_{\alpha j}\, d\xi$$

$$\overset{*}{\sigma}{}^{(n)}_{2j} = \int_{-1}^{1} \frac{dP(\xi)}{d\xi}\,\sigma_{2j}\, d\xi$$

By substituting the constitutive relation (II-2) for the displacement expansion (II-3) into the above approximate equations of motion, we can find governing equations of motion in terms of $u^{(0)}_1$, $u^{(0)}_2$, $u^{(0)}_3$, $u^{(1)}_1$... . The accuracy of this approximate theory depends on how many terms of the

displacement field we retain. Since the complexity in formulation increases rapidly with the number of terms included we keep terms only up to second order. Furthermore, we will examine harmonic waves propagating along the x_1 and x_2 directions so that we drop $u_3^{(n)}$ terms and set $\frac{\partial}{\partial x_3}\{\ \} = 0$. Next to get rid of the undesired coupling with higher modes we set $\ddot{u}_1^{(2)} = \ddot{u}_2^{(2)} = 0$. Then the resulting equations are

$$2b(c_{11}u_{1,11}^{(0)}+\frac{1}{b}c_{12}u_{2,1}^{(1)}) + (\sigma_{21}^{+}-\sigma_{21}^{-}) = 2b\rho\ddot{u}_1^{(0)}$$

$$2bc_{66}(\frac{1}{b}u_{1,1}^{(1)}+u_{2,11}^{(0)}) + (\sigma_{22}^{+}-\sigma_{22}^{-}) = 2b\rho\ddot{u}_2^{(0)}$$

$$\frac{2b}{3}(c_{11}u_{1,11}^{(1)}+\frac{3}{b}c_{12}u_{2,1}^{(2)}) - 2c_{66}(\frac{u_1^{(1)}}{b}+u_{2,1}^{(0)}) + (\sigma_{21}^{+}+\sigma_{21}^{-}) = \frac{2}{3}b\rho\ddot{u}_1^{(1)}$$

$$\frac{2b}{3}(c_{66}u_{2,11}^{(1)}+\frac{3}{b}c_{66}u_{1,1}^{(2)}) - 2(c_{12}u_{1,1}^{(0)}+\frac{1}{b}c_{22}u_2^{(1)}) + (\sigma_{22}^{+}+\sigma_{22}^{-}) = \frac{2}{3}b\rho\ddot{u}_2^{(1)}$$

$$(\sigma_{22}^{+}-\sigma_{22}^{-}) - 2(c_{12}u_{1,1}^{(1)}+\frac{3}{b}c_{22}u_2^{(2)}) = 0$$

$$(\sigma_{21}^{+}-\sigma_{21}^{-}) - 2(c_{66}u_{2,1}^{(1)}+\frac{3}{b}c_{66}u_1^{(2)}) = 0 \qquad \text{(II-5)}$$

where the sign + and - represent the stress components on the top and bottom surfaces of the layer under examination, i.e., at $\xi = \pm 1$. Here we notice that the first, fourth, and last equations are written in terms of $u_i^{(n)}$, where $(n+i)$ is an odd integer and represents the thickness stretching motion (or symmetric motion). In the rest of the equations in which $(n+i)$ is an even integer the displacements represent the flexual motion (or antisymmetric motion). Hence, this process has decoupled the

stretching motion from bending motion*. To get rid of the 2nd order modes from Eq. (II-5) we solve the last two equations for $u_2^{(2)}$ and $u_1^{(2)}$, and insert them into the remaining equations. Then Eq. (II-5) can be reduced as follows:

$$2b(c_{11}u_{1,11}^{(0)}+\frac{1}{b}c_{12}u_{2,1}^{(1)}) + (\overset{+}{\sigma}_{21}-\overset{-}{\sigma}_{21}) = 2b\rho\ddot{u}_1^{(0)}$$

$$2b\ c_{66}(\frac{1}{b}u_{1,1}^{(1)}+u_{2,11}^{(0)}) + (\overset{+}{\sigma}_{22}-\overset{-}{\sigma}_{22}) = 2b\rho\ddot{u}_2^{(0)} \qquad \text{(II-6)}$$

$$\frac{2b}{3}\hat{c}_{11}u_{1,1}^{(1)}-2c_{66}(\frac{u_1^{(1)}}{b}+u_{2,1}^{(0)}) + \frac{c_{12}b}{3c_{22}}(\overset{+}{\sigma}_{22}-\overset{-}{\sigma}_{22})_{,1} + (\overset{+}{\sigma}_{21}+\overset{-}{\sigma}_{21}) = \frac{2}{3}b\rho\ddot{u}_1^{(1)}$$

$$-2(c_{12}u_{1,1}^{(0)}+\frac{1}{b}c_{22}u_2^{(1)}) + (\overset{+}{\sigma}_{22}+\overset{-}{\sigma}_{22}) = \frac{2}{3}b\rho\ddot{u}_2^{(1)}$$

where $\hat{c}_{11} = c_{11} - c_{12}^2/c_{22}$.

Plate Analysis

In view of the Legendre polynomial expansion, the displacements on the both sides of a layer can be written as $\overset{\pm}{u}_i = u_i^{(0)} \pm u_i^{(1)}$ since the governing equations for a layer, Eq. (II-6), only include terms up to the first order of expansion, i.e., a linear expansion. Remembering that this analysis is valid for any arbitrary layer in a plate, say the nth layer, equation (II-6) can be immediately written as

---

* These two motions are, of course, coupled through the boundary conditions.

$$\rho(\ddot{u}_n+\ddot{u}_{n-1}) = c_{11}(u_n+u_{n-1})_{,11}+\frac{c_{12}}{b}(v_n-v_{n-1})_{,1}+\frac{1}{b}(\tau_n-\tau_{n-1})$$

$$\rho(\ddot{v}_n-\ddot{v}_{n-1}) = -\frac{3}{b}c_{12}(u_n+u_{n-1})_{,1}-\frac{3}{b^2}c_{22}(v_n-v_{n-1})+\frac{3}{b}(\sigma_n+\sigma_{n-1})+(\tau_n-\tau_{n-1})_{,1}$$

$$\rho(\ddot{u}_n-\ddot{u}_{n-1}) = \hat{c}_{11}(u_n-u_{n-1})_{,11}-\frac{3c_{66}}{b^2}(u_n-u_{n-1}) - \frac{3}{b}c_{66}(v_n+v_{n-1})_{,1}$$

$$+\frac{c_{12}}{c_{22}}(\sigma_n-\sigma_{n-1})_{,1} + \frac{3}{b}(\tau_n+\tau_{n-1}) \tag{II-7}$$

$$\rho(\ddot{v}_n+\ddot{v}_{n-1}) = c_{66}\ \{\frac{1}{b}(u_n-u_{n-1})_{,1}+(v_n+v_{n-1})_{,11}\} + \frac{1}{b}(\sigma_n-\sigma_{n-1})$$

where σ and τ are used to represent σ_{22} and σ_{12} and u and v denote u_1 and u_2, respectively. These equations are the approximate equations of motion of a layer written in the form of a difference-differential equation. For a plate made of N layer, the above equations contain $4(N+1)$ unknowns $(u_o,v_o,\tau_o,\sigma_o,\ldots u_N,v_N,\tau_N,\sigma_N)$ and offer 4N equations. Since the additional four conditions are supplied by boundary conditions on the top and bottom surfaces, solutions of these equations can be found.

In Eq. (II-7) we notice some important points. The first point is that the logitudinal coordinate x_1 and the time variable are continuous variables while the thickness coordinate x_2 is now discrete. This enables us to use integral transforms in x_1 and time variables so that we can arrive at pure difference equations after integral transforms. The second point concerns the continuity conditions of stress and displacement. We note that u, v, σ_{22}, and σ_{12} have to be continuous across the layer boundary and these conditions are identically satisfied by

Eq. (II-7). But the normal stress tangential to the layer boundary is not necessarily continuous and Eq. (II-7) allows such a possibility. One can retain higher order terms in the displacement expansion given by Eq. (II-3) to give more accurate results. This can be achieved more easily by using Eq. (II-7) and increasing the number of layers in a plate under consideration. This process does not give any additional difficulties except a little more computer time.

2. Dispersion Relationships of Harmonic Waves

Harmonic Waves

Before we examine the transient propagation of stress wave due to an impact we first investigate dispersion relations of harmonic waves in a composite plate governed by approximate equations of motion (II-7). For harmonic waves propagating along the x_1 axis we assume

$$\{u_n, v_n, \sigma_n, \tau_n\} = \{U_n, V_n, \Sigma_n, T_n\}\, e^{i(kx_1-\omega t)} \tag{II-8}$$

Substituting this into the approximate equations of motion (II-7) we obtain

$$(\bar{\omega}^2 - c_{11}\kappa^2)(U_n+U_{n-1}) + c_{12}i\kappa(V_n-V_{n-1}) + b(T_n-T_{n-1}) = 0$$

$$-3c_{12}i\kappa(U_n+U_{n-1}) + (\bar{\omega}^2-3c_{22})(V_n-V_{n-1}) + i\kappa b(T_n-T_{n-1}) + 3b(\Sigma_n+\Sigma_{n-1}) = 0$$

$$3c_{66}i\kappa(U_n-U_{n-1}) + (\bar{\omega}^2-c_{66}\kappa^2)(V_n+V_{n-1}) + b(\Sigma_n-\Sigma_{n-1}) = 0 \tag{II-9}$$

$$(\bar{\omega}^2-\hat{c}_{11}\kappa^2-3c_{66})(U_n-U_{n-1}) - 3c_{66}i\kappa(V_n+V_{n-1}) + 3b(T_n+T_{n-1})$$

$$+\frac{c_{12}}{c_{12}}\, i\kappa b(\Sigma_n-\Sigma_{n-1}) = 0$$

for $n = 1, 2 \ldots N$. Here we set

$$\kappa = bk = k\left(\frac{\Delta}{2N}\right), \quad \Delta = 2bN$$

$$\bar{\omega}^2 = \rho b^2 \omega^2 = \rho\omega^2 \left(\frac{\Delta}{2N}\right)^2$$

and Δ is the total thickness of the plate. For a plate consisting of N layers, the boundary conditions require traction free surfaces, namely, $T_o = \Sigma_o = T_N = \Sigma_N = 0$. When these conditions are applied to equation (II-9) we obtain 4N equations in terms of 4N unknowns (U_o, V_o; U_n, V_n, T_n, Σ_n with $n = 1, \ldots N-1$; U_N, V_N). By setting the coefficient matrix to be singular, required dispersion relationships can be obtained.

<u>One-layer Plate</u>

The dispersion relationship for a plate made of a single layer can be found by setting $N = 1$ in equation (II-9) with $\Sigma_o = T_o = \Sigma_1 = T_1 = 0$. The resulting equations are now written in matrix form as follows:

$$\begin{bmatrix} (\bar{\omega}^2 - c_{11}\kappa^2) & c_{12}i\kappa & 0 & 0 \\ -c_{12}i\kappa & \frac{1}{3}(-3c_{22}+\bar{\omega}^2) & 0 & 0 \\ 0 & 0 & c_{66}i\kappa & c_{66}\kappa^2 - \bar{\omega}^2 \\ 0 & 0 & \hat{c}_{11}\kappa^2 + 3c_{66} - \bar{\omega}^2 & 3c_{66}i\kappa \end{bmatrix} \cdot \begin{bmatrix} U_1 + U_o \\ V_1 - V_o \\ U_1 - U_o \\ V_1 + V_o \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{II-10}$$

Then by setting the determinant of the coefficient matrix to zero we obtain

$$c_{11}\kappa^2 - \frac{1}{3}(\bar{\omega}^2-3c_{22})(\bar{\omega}^2-c_{11}\kappa^2) = 0$$

$$c_{66}\kappa^2 - \frac{1}{3}(\bar{\omega}^2-\hat{c}_{11}\kappa^2-3c_{66})(\bar{\omega}^2-c_{66}\kappa^2) = 0 \quad . \tag{II-11}$$

Here we notice that the first relationship corresponds to the state of deformation of $U_1 = U_o$ and $V_1 = -V_o$, which represents the thickness extension of the plate (or the symmetric mode), and the second describes the flexual deformation (or antisymmetric mode). The exact theory of plates gives an infinite number of dispersion relationships, but because this model only has two inertia points (namely n = 0, 1), each of them having two components of displacement, we only have the first four relationships.

Dispersion relationships and corresponding phase velocities for an isotropic plate with Poisson's ratio 1/4 (namely $\lambda = \mu$) are given in Fig. 2a and 2b up to the range where the wave length becomes equal to the plate thickness. Solid lines represent the symmetric modes and dotted lines the antisymmetric modes. As predicted by Mindlin and Medick the optical branch of the symmetric mode approaches the dilatation wave [18]. The acoustic branch of the antisymmetric mode starts from the bending motion and approaches the shear wave when the wave number k becomes larger and larger*. Similar relationships for an anisotropic plate made of 55% graphite fiber-epoxy matrix with a layup angle of 45° are shown in Fig. 3a and 3b.

---

* See section 5 for discussion about the large wave number limit.

Two-layer Plate

In this case we obtain eight equations by putting n = 1 and 2 in equation (II-9). Boundary conditions require $T_o = \Sigma_o = T_2 = \Sigma_2 = 0$. By following the same procedure we find the dispersion relations as

$$\{(\bar{\omega}^2 - c_{11}\kappa^2)(\bar{\omega}^2 - 3c_{22}) - 3c_{12}^2\kappa^2\}(\bar{\omega}^2 - \hat{c}_{11}\kappa^2 - 3c_{66} + \frac{c_{66}c_{12}}{c_{22}}\kappa^2)$$
$$+3(\bar{\omega}^2 - c_{11}\kappa^2)\{(\bar{\omega}^2 - c_{66}\kappa^2)(\bar{\omega}^2 - \hat{c}_{11}\kappa^2 - 3c_{66}) - 3c_{66}^2\kappa^2\} = 0$$

(II-12)

$$\{(\bar{\omega}^2 - c_{66}\kappa^2)(\bar{\omega}^2 - \hat{c}_{11}\kappa^2 - 3c_{66}) - 3c_{66}^2\kappa^2\}$$
$$+3(\bar{\omega}^2 - c_{66}\kappa^2)\{(\bar{\omega}^2 - c_{11}\kappa^2)(\bar{\omega}^2 - 3c_{22}) - 3c_{12}^2\kappa^2\} = 0$$

Again the first equation represents the symmetric mode and is shown as solid lines in Fig. 4 and 5. The second equation is plotted with dotted lines representing the antisymmetric mode.

As expected we have six relationships since the this two-layer model is equivalent to a three-mass system with two degrees of freedom for each mass. When the wave number $k\Delta$ increases the following are observed: for the symmetric mode the upper optical branch approaches the dilatation wave, whereas for the antisymmetric mode the lower optical branch approaches the shear wave*.

---

* See section 5 for discussions about the large wave number limit.

N-Layer Plate

In general, we can obtain a $2(N+1)$ order polynomial of $\bar{\omega}^2$ by expanding a $(4N) \times (4N)$ determinant and finding $2(N+1)$ dispersion relationships. But, unfortunately, this process involves considerably complicated algebra and it may be necessary to develop a computer technique to find roots of an equation in determinant form (not in polynomial form).

A difference equation approach can be used to solve the N set of four simultaneous first order difference equations given by Eq. (II-9). This proceedure is neat and can be generalized for any number of layers as discussed in the next section; but the last step of this approach, where a long polynomial is to be solved again, is not any simpler than the previous direct method.

3. Impact on an Elastic Composite Plate

Normalization and Integral Transforms of Governing Equations

The governing equations given by (II-7) are first nondimensionalized as follows:

$$\{U_n, V_n, \eta\} = \{u_n/\Delta, U_n/\Delta, x_1/\Delta\}$$

$$\{C_{ij}, T_n, \Sigma_n\} = \{c_{ij}/c_{66}, \tau_n/c_{66}, \sigma_n/c_{66}\}$$

$$\tau = t/T_o$$

where Δ is the total thickness of the plate and T_o is the time required for the quasi-shear wave to travel the impact radius. Next we apply a Laplace transform in τ and a Fourier transform in η, i.e.,

$$\hat{g}(s) = \int_o^\infty g(t)e^{-s\tau}d\tau$$

$$\bar{g}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(\eta)e^{ik\eta}d\eta \quad .$$

Then the resulting equations are

$$-(fs^2+\frac{C_{11}}{2N}k^2)(\hat{\bar{U}}_n+\hat{\bar{U}}_{n-1})-C_{12}ik(\hat{\bar{V}}_n-\hat{\bar{V}}_{n-1})+(\hat{\bar{T}}_n-\hat{\bar{T}}_{n-1}) = 0$$

$$C_{12}ik(\hat{\bar{U}}_n+\hat{\bar{U}}_{n-1})-(\frac{fs^2}{3}+2NC_{22})(\hat{\bar{V}}_{n-1}-\hat{\bar{V}}_{n-1})+(\hat{\bar{\Sigma}}_n+\hat{\bar{\Sigma}}_{n-1})-\frac{ik}{6N}(\hat{\bar{T}}_n-\hat{\bar{T}}_{n-1}) = 0$$

$$-C_{66}ik(\hat{\bar{U}}_n-\hat{\bar{U}}_{n-1})-(fs^2+\frac{C_{66}}{2N}k^2)(\hat{\bar{V}}_n+\hat{\bar{V}}_{n-1})+(\hat{\bar{\Sigma}}_n-\hat{\bar{\Sigma}}_{n-1}) = 0 \tag{II-13}$$

$$-(\frac{fs^2}{3}+\frac{\hat{C}_{11}k^2}{6N}+2NC_{66})(\hat{\bar{U}}_n-\hat{\bar{U}}_{n-1})+C_{66}ik(\hat{\bar{V}}_n+\hat{\bar{V}}_{n-1})-\frac{C_{12}ik}{6NC_{22}}(\hat{\bar{\Sigma}}_n-\hat{\bar{\Sigma}}_{n-1})+(\hat{\bar{T}}_n-\hat{\bar{T}}_{n-1}) = 0$$

where the normalization factor f is given as

$$f = \frac{1}{2N}\frac{\Delta^2\rho}{c_{66}T_o^2} = \frac{b\Delta\rho}{c_{66}T_o^2} .$$

Solution of Difference Equations

Since the simultaneous difference equations given are linear and all the coefficients are constants the solution [26] has to be

$$\{\hat{\bar{U}}_n, \hat{\bar{V}}_n, \hat{\bar{T}}_n, \hat{\bar{\Sigma}}_n\} = \{A,B,C,D\}e^{2i\theta n} \qquad \text{(II-14)}$$

where the phase shift θ is complex, in general, and propagation through the thickness direction in the plate is characterized by θ. Namely, θ is the wave number in the thickness direction. By substituting solution (II-14) into the difference equation (II-13) we obtain a set of four simultaneous homogeneous equations through which the relationships among the constants A,B,C, and D have to be determined. If we set the resulting coefficient matrix of A,B,C, and D to be singular we obtain the following equation for phase shift θ:

$$\cos^4\theta\left(fs^2+\frac{C_{11}k^2}{2N}\right)\left(fs^2+\frac{C_{66}}{2N}k^2\right)$$

$$+\sin^4\theta\left(\frac{fs^2}{3}+2NC_{22}-\frac{C_{12}k^2}{6N}\right)\cdot\left(\frac{fs^2}{3}+\frac{\hat{C}_{11}}{6N}k^2+2NC_{66}-\frac{C_{66}C_{12}}{6NC_{22}}k^2\right)$$

$$+\cos^2\theta\sin^2\theta\left[\left(fs^2+\frac{C_{11}}{2N}k^2\right)\left\{\frac{k^2}{6N}C_{66}+\frac{fs^2}{3}+2NC_{22}-\left(\frac{k}{6N}\right)^2\frac{C_{66}}{C_{22}}\left(fs^2+\frac{C_{66}}{2N}k^2\right)\right\}\right.$$

$$\left.-(C_{12}+C_{66})^2k^2+\left(fs^2+\frac{C_{66}}{2N}k^2\right)\left(\frac{fs^2}{3}+\frac{\hat{C}_{11}}{6N}k^2+2NC_{66}+\frac{C_{12}^2k^2}{6NC_{22}}\right)\right] \qquad \text{(II-15)}$$

$$= a_1\cos^4\theta + a_2\cos^2\theta + a_3 = 0 .$$

This equation implies that for a given wave number k along x_1 and a frequency s (s represents the frequency for the case of harmonic waves), an infinite value of wave numbers exists for propagation through the thickness direction,but only four of them are sufficient to give all linearly independent solutions of the form of Eq. (II-14). If we denote the solution of the phase shift equation as

$$\cos^2\beta = \frac{-a_2+\sqrt{a_2^2-4a_1a_3}}{2a_1}$$

$$\cos^2\alpha = \frac{-a_2-\sqrt{a_2^2-4a_1a_3}}{2a_1} \tag{II-16}$$

the complete general solutions of difference equation (II-13) are

$$\begin{bmatrix} \hat{\hat{U}}_n \\ \hat{\hat{V}}_n \\ \hat{\hat{T}}_n \\ \hat{\hat{\Sigma}}_n \end{bmatrix} = \begin{bmatrix} A_1 \\ B_1 \\ C_1 \\ D_1 \end{bmatrix} e^{2i\beta n} + \begin{bmatrix} A_2 \\ B_2 \\ C_2 \\ D_2 \end{bmatrix} e^{-2i\beta n} + \begin{bmatrix} A_3 \\ B_3 \\ C_3 \\ D_3 \end{bmatrix} e^{2i\alpha n} + \begin{bmatrix} A_4 \\ B_4 \\ C_4 \\ D_4 \end{bmatrix} e^{-2i\alpha n} . \tag{II-17}$$

Next,by substituting the above solutions into the original difference equations (II-13) we find the relationships among A_i, B_i, C_i, and D_i. The results are

$$\begin{bmatrix} \hat{\hat{U}}_n \\ \hat{\hat{V}}_n \\ \hat{\hat{T}}_n \\ \hat{\hat{\Sigma}}_N \end{bmatrix} = \begin{bmatrix} X_1(\beta)\ E_1 \\ X_2(\beta)\ E_2 \\ X_3(\beta)\ E_3 \\ E_1 \end{bmatrix} \cdot \cos 2n\beta + i \begin{bmatrix} X_1(\beta)\ E_2 \\ X_2(\beta)\ E_1 \\ X_3(\beta)\ E_1 \\ E_2 \end{bmatrix} \cdot \sin 2n\beta$$

$$+ \begin{bmatrix} Y_1(\alpha)\ E_4 \\ Y_2(\alpha)\ E_3 \\ E_3 \\ Y_3(\alpha)\ E_4 \end{bmatrix} \cdot \cos 2n\alpha + i \begin{bmatrix} Y_1(\alpha)\ E_3 \\ Y_2(\alpha)\ E_4 \\ E_4 \\ Y_3(\alpha)\ E_3 \end{bmatrix} \cdot \sin 2n\alpha \qquad \text{(II-18)}$$

where $E_1 = D_1 + D_2$, $E_2 = D_1 - D_2$, $E_3 = C_3 + C_4$, $E_4 = C_3 - C_4$ and

$$X_i(\beta) = -\frac{\Delta_i(\beta)}{\Delta(\beta)}$$

$$\Delta(\beta) = (fs^2 + \frac{C_{11}}{2n}k^2)(fs^2 + \frac{C_{66}}{2n}k^2)\cos^3\beta$$

$$+ \{(fs^2 + \frac{C_{66}}{2n}k^2)(\frac{fs^2}{3} + \frac{\hat{C}_{11}}{6n}k^2 + 2nC_{66}) - C_{66}C_{12}k^2 - C_{66}^2k^2\}\sin^2\beta\cdot\cos\beta$$

$$\Delta_1(\beta) = ik\ \sin^2\beta\cos\beta\{\frac{C_{12}}{6nc_{22}}(fs^2 + \frac{C_{66}}{2n}k^2) - (C_{12} + C_{66})\} \qquad \text{(II-19)}$$

$$\Delta_2(\beta) = i\ \sin^3\beta\{\frac{C_{66}C_{12}}{6nc_{22}}k^2 - (\frac{fs^2}{3} + \frac{\hat{C}_{11}}{6n}k^2 + 2nC_{66})\} - \cos^2\beta\sin\beta(fs^2 + \frac{C_{11}}{2n}k^2)$$

$$\Delta_3(\beta) = k\sin^3\beta\{(\frac{fs^2}{3} + \frac{\hat{C}_{11}}{6n}k^2 + 2nC_{66})C_{12} - \frac{C_{12}^2k^2}{6nC_{22}}C_{66}\}$$

$$+ k\ \sin\beta\cos^2\beta\{\frac{C_{12}}{6nC_{22}}(fs^2 + \frac{C_{11}k^2}{2n})\cdot(fs^2 + \frac{C_{66}}{2n}k^2) - (fs^2 + \frac{C_{11}}{2n}k^2)C_{66}\}$$

and

$$Y_i(\alpha) = -\frac{\bar{\Delta}_i(\alpha)}{\bar{\Delta}(\alpha)}$$

$$\bar{\Delta}(\alpha) = i\cos^2\alpha\cdot\sin\alpha\{C_{66}k^2(C_{12}+C_{66})-(fs^2+\frac{C_{66}}{2n}k^2)\cdot(\frac{fs^2}{3}+\frac{\hat{C}_{11}k^2}{6n}$$

$$+ 2nC_{66}+\frac{C_{12}^2k^2}{6nC_{22}})\} + i\sin^3\alpha(\frac{fs^2}{3}+2nC_{22})$$

$$\times(\frac{C_{66}C_{12}k^2}{6nC_{22}}-\frac{fs^2}{3}-\frac{C_{11}k^2}{6n}-2nC_{66})$$

$$\bar{\Delta}_1(\alpha) = \cos^3\alpha(fs^2+\frac{C_{66}}{2n}k^2)$$

$$+\sin^2\alpha\cdot\cos\alpha\{-(\frac{k}{6n})\frac{2C_{12}}{C_{22}}(fs^2+\frac{C_{66}}{2n}k^2)+\frac{k^2}{6n}C_{66}+(\frac{fs^2}{3}+2nC_{22})\}$$

$$\bar{\Delta}_2(\alpha) = k\cdot\cos^2\alpha\sin\alpha(C_{12}+C_{66}) \qquad \text{(II-20)}$$

$$+ k\sin^3\alpha\{\frac{1}{6n}(\frac{fs^2}{3}+\frac{\hat{C}_{11}k^2}{6n}+2nC_{66})-(\frac{k}{6n})^2\frac{C_{12}C_{66}}{C_{22}}\}$$

$$\bar{\Delta}_3(\alpha) = ik\sin^2\alpha\cos\alpha\{\frac{C_{66}^2}{6n}k^2-\frac{1}{6n}(\frac{fs^2}{3}+\frac{\hat{C}_{11}k^2}{6n}+2nC_{66})(fs^2+\frac{C_{66}}{2n}k^2)$$

$$+C_{66}k(\frac{fs^2}{3}+2nC_{22})\}+ik\cos^3\alpha\{-C_{12}(fs^2+\frac{C_{66}}{2n}k^2)\}.$$

Equations (II-18) with (II-19,20) and the phase shifts α and β given by (II-16) constitute the final form of the general solutions of the difference equations (II-13).

In Eqs. (II-19,20) we notice that when $k \to 0$ we have $X_1(\beta) = X_3(\beta) = Y_2(\alpha) = Y_3(\alpha) = 0$. Namely the propagation of the normal stress (with phase shift β) and the propagation of the shear stress (with phase shift α) are completely decoupled. This occurs when the waves are propagating only through the thickness direction [27].

Impact Boundary Condition

Boundary conditions for an impact can be described by any two conditions among u_o, v_o, σ_o, and τ_o and another two conditions from u_N, v_N, σ_N, and τ_N. For our present problem we have chosen a line impact by a normal stress along the x_3 axis (Figure 1), i. e.,

$$\sigma_o = -\frac{P_o}{4}\left(1-\cos\frac{2\pi t}{t_o}\right)\left(1+\cos\frac{\pi x}{a}\right) : \quad |x| \leq a \quad \text{and} \quad 0 \leq t \leq t_o$$

$$= 0 : \quad |x| > a \quad \text{or} \quad t > 0 \quad \text{or} \quad t > t_o$$

$$\tau_o = \sigma_N = \tau_N = 0 \ . \qquad \text{(II-21)}$$

Hence, the boundary conditions for the present impact problem lead to the following equation

$$\begin{bmatrix} 0 & , & X_3(\beta) & , & 1 & , & 0 \\ 1 & , & 0 & , & 0 & , & Y_3(\alpha) \\ iX_3(\beta)\sin 2\beta N & , & X_3(\beta)\cos 2\beta N & , & \cos 2\alpha N & , & i\sin 2\alpha N \\ \cos 2\beta N & , & i\sin 2\beta N & , & iY_3(\alpha)\sin 2\alpha N & , & Y_3(\alpha)\cos 2\alpha N \end{bmatrix} \cdot \begin{bmatrix} E_1 \\ E_3 \\ E_3 \\ E_4 \end{bmatrix} = \begin{bmatrix} 0 \\ q \\ 0 \\ 0 \end{bmatrix} \qquad \text{(II-22)}$$

where q is the integral transform of the impact function (II-21)*. Solving the above equations for E_i's we can have

$$\begin{aligned} E_i &= \frac{\mathcal{D}_i}{\mathcal{D}}\, q \\ \mathcal{D} &= \{1+X_3^2(\beta)Y_3^2(\alpha)\}\sin 2\alpha N\cdot\sin 2\beta N + 2X_3(\beta)Y_3(\alpha)(\cos 2\alpha N\cdot\cos 2\beta N-1) \\ \mathcal{D}_1 &= X_3(\beta)Y_3(\alpha)(\cos 2\alpha N\cdot\cos 2\beta N-1) + \sin 2\alpha N\cdot\sin 2\beta N \\ \mathcal{D}_2 &= i\{\cos 2\beta N\cdot\sin 2\alpha N - X_3(\beta)Y_3(\alpha)\cos 2\alpha N\cdot\sin 2\alpha N\} \qquad \text{(II-23)} \\ \mathcal{D}_3 &= iX_3(\beta)\{X_3(\beta)Y_3(\alpha)\sin 2\beta N\cdot\cos 2\alpha N - \cos 2\beta N\cdot\sin 2\alpha N\} \\ \mathcal{D}_4 &= X_3(\beta)\{X_3(\beta)Y_3(\alpha)\sin 2\alpha N\cdot\sin 2\beta N + \cos 2\beta N\cdot\cos 2\alpha N-1\}\ . \end{aligned}$$

Substituting the E_i's into the general solution (II-18) we can find the complete solutions which satisfy the impact boundary conditions given by (II-21). In other words, for given values of k and s we first find the phase shift α and β from (II-15,16) and with these we can find solutions in integral transform from equations (II-18,23) which are the final solutions under impact. After $\hat{\bar{U}}_n$, $\hat{\bar{V}}_n$, $\hat{\bar{T}}_n$ and $\hat{\bar{\Sigma}}_n$ are calculated

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* Allowing the determinant of the coefficient matrix to vanish leads to dispersion relations of an N-layer plate, namely $\mathcal{D}(\alpha,\beta) = 0$. Then, α and β are obtained from (II-15,16) which gives the complete dispersion relationships.

with a given impact function q, they can be inverted easily by means of the fast Fourier transform routine [3,20] to give the complete displacement and the stress fields after impact.

Tangential Normal Stress

As discussed following Eq. (II-7), the tangential normal stress does not appear explicitly in the approximate equations of motion. Therefore, this component of the stress has to be calculated from the constitutive equation. Namely,

$$\sigma_{11_o} = c_{11}u_{o,1} + \frac{c_{12}}{2b}(v_1 - v_o)$$

$$\sigma_{11_n} = c_{11}u_{n,1} + \frac{c_{12}}{2b}(v_n - v_{n-1}) \qquad 1 \le n \le N$$

or after normalization and integral transform they are

$$\hat{\bar{\sigma}}_{11_o} = -ikC_{11}\hat{\bar{U}}_o + N \cdot C_{12}(\hat{\bar{V}}_1 - \hat{\bar{V}}_o) \qquad \text{(II-24)}$$

$$\hat{\bar{\sigma}}_{11_n} = -ikC_{11}\hat{\bar{U}}_n + N \cdot C_{12}(\hat{\bar{V}}_n - \hat{\bar{V}}_{n-1}) \qquad 1 \le n \le N \ .$$

Then once the displacement field is computed the tangential normal stress can be obtained from the above equation and inverted.

4. Some Numerical Results

The analysis discussed in the previous section includes the transient propagation in all directions but suitable choices of impact time, impact radius, sizes of time and distance steps are essential to make good use of the fast Fourier transform. For example, if we take a large time increment with a relatively thin plate propagation through the thickness will not be seen. For this matter we have examined several different cases.

Case 1: Longitudinal propagation

Propagation of impact generated waves along the longitudinal direction is examined for an isotropic plate (steel plate: $\lambda = \mu = 1.2\times10^7$ psi) employing a two-layer model. For these calculations we used an impact time $t_o = 10$ μsec, plate thickness $\Delta = 1$ cm, and impact radius $a = 4$ cm. Some of the results at a few different time sequences are shown in Fig. 6 a-f.

In these figures we can see two distinct states of propagation and corresponding wave fronts: one for horizontal displacements (u) and longitudinal normal stresses (σ_{11}), and another for vertical displacements (v) and shear stresses (τ). In other words, the initial signals of the horizontal displacements and longitudinal normal stresses propagate through the plate with longitudinal wave speed at amplitudes that are relatively small. But the major parts of their signals are due to a bending wave propagating with shear stresses and vertical displacements with a lower velocity. When the group velocities of these waves are calculated from the numerical results, they are about 5 mm/μsec and 3 mm/μsec, respectively, while the phase velocities of the unbounded

medium of this material are $C_d = \sqrt{(\lambda+2\mu)/\rho} = 5.61$ mm/μsec and $C_s = \sqrt{\mu/\rho} = 3.25$ mm/μsec.

Case 2: Propagation Through Thickness

To examine the propagation through the thickness it is necessary to have a sufficient number of layers in a plate. It is also essential to make the time step relatively small compared to the layer thickness. To do this we increase the thickness of the plate and the number of layers and reduce the impact time.

In Figs.7 and 8 propagation of the transverse normal stress in the same plate ($\Delta = 4$ cm, $t_o = 2$μsec, $a = 40$ cm; 4-layer model) is shown at different time sequences. As seen in Fig. 7, the transverse normal stress is initially compressive due to the impact and a compression wave propagates through the thickness. But later it becomes a tension wave after reflection from the free surface and the tension wave propagates back to the impact surface. In Fig. 8 we see the change of the transverse normal stress and the interlaminar shear stress with time for the same impact conditions as in Fig. 7.

Similar results are also shown for the case of an anisotropic plate in Fig. 9 (55% graphite fibers-epoxy matrix, layup angle = 15°; $\Delta = 1$ cm, $t_o = 2$μsec, $a = 2$ cm; 8-layer model). Here we again notice a clear delay in time for waves to travel from one layer to the next one. Another important point is that the shape of the impact stress is relatively well preserved during the initial stage of propagation but changes immediately afterwards. The distortion of the shape becomes more serious with further propagation due to reflection, thus, showing the highly dispersive nature of the harmonic waves in the approximate plate theory.

When the group velocities are calculated from these results, we find 6.32 mm/μsec for the dilatation wave and 3.33 mm/μsec for the shear wave in the case of the isotropic plate and 2.5 mm/μsec for the quasi-dilatation wave of the anisotropic plate. Their expected values are, respectively, 5.61, 3.25, and 2.36 mm/μsec. In other words, waves going through the thickness are traveling faster than expected.

Case 3. Wave Surfaces

In the previous two cases we examined the transient waves propagating dominantly along either the x_1 axis or through the thickness direction by suitable choices of the plate geometry and impact condition.
now examine the combined effect, simultaneous propagation in both directions. This effect is shown in Fig. 10 (isotropic plate; Δ = 4 cm, t_o = 4μsec, a = 4 cm; 4-layer model) where the transverse normal stress generated from the line source due to impact not only spreads out in all directions but also reflects from the free surface.

When the plate is anisotropic, the situation is more complex in the sense that waves are neither dilatation nor shear but they are coupled together (now called quasi-dilational or quasi-shear waves). Due to the coupling, phase velocities of the anisotropic wave vary from one direction to another, resulting in complicated shapes for the velocity surfaces and wave fronts [2]. For an ansiotropic plate (made of 55% graphite fiber-epoxy matrix with layup angle 45°) the velocity surfaces and the wave surfaces are shown in Fig. 11. The stress state at 10 μsec after the impact on the same plate (Δ = 4 cm, t_o = 4μsec, a = 2 cm; 8-layer model) with the corresponding wave fronts are shown in Fig. 12a. In the

propagation of the quasi-longitudinal wave we notice that the longitudinal propagation is well bounded by the quasi-dilatational wave surface but the transverse propagation is not. The shear wave is not bounded by the quasi-shear wave front in either direction.

This interesting phenomenon of higher propagation speeds through the thickness is related to the dispersion relationship at short wave length limits; it is discussed in the next section.

5. Correction Factor and Conclusion

Correction Factor

According to the present model of a multilayer plate, one of the antisymmetric modes of the dispersion relationships approaches the shear speed when the wave length becomes shorter and shorter, as mentioned in Section 2. It is well understood that such a limit is incorrect, i.e., in the limit of short wave length there should be a Rayleigh wave instead of a shear wave. Such a discrepancy can be eliminated by introduction of proper correction factors, as shown by Mindlin and Medick [18]. Correction factors can be found by examining either the large wave number limit or the cut-off frequencies of both the exact theory and the present approximate theory. Since these two ways lead us to the same results we will find the factors by matching the cut-off frequencies of the two theories.

The cut-off frequencies of the exact theory for an isotropic plate can be obtained from the well-known Rayleigh-Lam's equation. The lowest cut-off frequencies are $\frac{\pi}{\Delta}\sqrt{(\lambda+2\mu)/\rho}$ for the symmetric mode and $\frac{\pi}{\Delta}\sqrt{\mu/\rho}$ for the antisymmetric mode. The corresponding cut-off frequencies of our approximate theory are $\frac{2}{\Delta}\sqrt{3c_{22}/\rho}$ and $\frac{2}{\Delta}\sqrt{3c_{66}/\rho}$*. Hence, we notice that replacing c_{22} by $c_{22}\pi^2/12$ and c_{66} by $c_{66}\pi^2/12$ makes the two theories have the same two lowest cut-off frequencies. Furthermore the shear wave observed in the short wave length limit of the present approximation becomes a wave with a speed of $\frac{\pi}{\sqrt{12}}\sqrt{\mu/\rho}$, i.e., the Rayleigh wave.

Another important consequence of the correction factor is to reduce propagation speeds through the thickness, which are related to $\sqrt{c_{22}/\rho}$

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* The lowest two cut-off frequencies are found from Eq. (II-11) and they are independent of the layer number in the plate under investigation.

and $\sqrt{c_{66}/\rho}$, with a factor of $\pi/\sqrt{12}$. Propagation of the maximum value of the interlaminar normal stress through the thickness is examined with and without correction factors and the results are shown in Fig. 13. Without the correction factor the propagation speed in a composite plate is roughly about 2.60 mm/μsec obtained from the numerical results used in Fig. 12. When the same plate is subjected to identical impact conditions this reduces to about 2.41 mm/μsec with the correction factor. Comparing this with the group velocity in an unbounded composite space (= 2.36 mm/μsec) the agreement of the present approximate theory is remarkable. Similar results are also observed in the case of shear and quasi-shear waves. When these correction factors are introduced in the previous cases, shown in Figs. 8, 9, and 12, all the signals propagating through the thickness are now well bounded within the corresponding wave fronts, as shown in Fig. 12b and from this we can notice the importance of the correction factors.

Discussion and Computation Time

It is interesting to compare the computation time of this model with some other methods, such as the finite element method or the finite difference method. In the case of an 8-layer anisotropic plate model, from which Figs. 9 and 12 are produced, we have

9 steps along the thickness: 8-layer model;

32 step along the x_1 direction: 64 points are used in pratice but only half of them are useful because of the symmetry of the problem,

32 steps in time;

2 displacement components at each point.

Therefore the total number of the unknowns, which are the basic unknowns either in case of the finite difference or finite element methods, is

18,432. After these primary variables are calculated, 27,648 secondary variables (three stress components at each points) have to be calculated again. According to our present model all these processes require only 200 K of computer space without using magnetic tapes or any kind of additional storage space and only 1 minute 6 seconds for CPU time in the IBM 370-168 model including compiling, linkage editing, I/O and execution.

Conclusion

The present theory is a generalization of Mindlin's approximate plate theory applied to a multilayer plate under an impact. By combined use of the finite difference technique in the thickness direction and the fast Fourier transform in the plane of plate and time, this model can be very useful for the study of wave propagation in a composite plate under impact forces. However, reasonable attention in usage of the fast Fourier transform is required to avoid spurious data. From the limited numerical data obtained from this model it appears that the anisotropy in the plate will lead to a considerable interlaminar shear which might lead to ply bonding failures. The model also shows that for short enough impact times, an interlaminar tension can develop as one would expect, which might also account for interlaminar ply failure.

III. IMPACT OF A COMPOSITE PLATE WITH AN INTERLAMINAR DAMPING LAYER

1. Description of Problem

Geometry of Plate

As an extension of the multilayer plate discussed in Chapter II we now examine the impact and the consequent stress wave propagation in a composite plate with viscoelastic damping layers. Possible models for damping mechanisms in plates are shown in Fig. 14. We will formulate a model made of an alternating number of elastic and viscoelastic layers, as shown in Fig. 14-c. As long as the layer structure of the plate is periodic, the main part of the analysis in Chapter II for an elastic plate is valid with additional equations for viscoelastic layers.

Viscoelastic Property of Elastomer

The mechanical properties of an elastomer are usually expressed in terms of a complex modulus depending on the frequency, i.e.,

$$G^{*}(\omega) = G'(\omega) + iG''(\omega) \ . \tag{III-1}$$

With this the constitutive equation is written as

$$\bar{\sigma}_{ij}(\omega) = G^{*}(\omega)\bar{\varepsilon}_{ij}(\omega) \tag{III-2}$$

in the frequency space where $\bar{\sigma}_{ij}(\omega)$ and $\varepsilon_{ij}(\omega)$ are respectively the Fourier transforms of σ_{ij} and ε_{ij} in time* [29].

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* For (III-2) the Fourier transform is defined as

$$\bar{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt \ .$$

The constitutive relation (III-2) with the complex modulus (III-1) implies the following constitutive equation in a time space:

$$\underset{\sim}{\sigma}(x,t) = \int_{-\infty}^{t} G(t-\tau)\dot{\underset{\sim}{\varepsilon}}(x,\tau)d\tau \qquad \text{(III-3)}$$

where the relaxation function $G(t)$ is related with the complex modulus $G^*(\omega)$ as

$$G^*(\omega) = \frac{1}{i\omega}\int_{0}^{\infty} G(t)e^{-i\omega t}dt \ . \qquad \text{(III-4)}$$

Therefore, when the complex modulus $G^*(\omega)$ is obtained by experiments, usually by means of harmonic excitation of strain, the relaxation function $G(t)$ can be found by inversion of equation (III-4).

The viscoelastic property of the elastomer under consideration has been extensively investigated (e.g. [19]) and its complex modulus is given in Fig. 15. This complex modulus can be reasonably well described by a three parameter equation as

$$G'(\omega) = a - \frac{6}{\omega^2+c} \ . \qquad \text{(III-5)}$$

These three parameters are obtained from another set of parameters: the maximum values of $G'(\omega)$ when $\omega\to\infty$, the maximum value of $G''(\omega)/G'(\omega)$ and the ω_o at which $G''(\omega)/G'(\omega)$ becomes the maximum. Therefore, if we characterize the complex modulus by proper choice of $G'(\omega)$, ω_o and the maximum of $G''(\omega_o)/G'(\omega_o)$, the relaxation functions are completely described.

2. Formulation

Elastic Layer

In Fig. 16 a typical viscoelastic layer (nth) is shown between two adjacent elastic layers (nth and (n+1)th) with appropriate discretization. The approximate equations of motion for the nth elastic layer given by Eq. (II-7) are still valid. But remembering the new discretizing notation in Fig. 16 we now have to replace $(\ )_n$ and $(\ )_{n-1}$ by $(\ )_n^-$ and $(\ )_{n-1}^+$, respectively. The results are

$$
\begin{aligned}
\rho(\ddot{u}_n^- + \ddot{u}_{n-1}^+) &= c_{11}(u_n^- + u_{u-1}^+)_{,11} + \frac{c_{12}}{b}(v_n^- - v_{n-1}^+)_{,1} + \frac{1}{b}(\tau_n^- - \tau_{n-1}^+) \\
\rho(\ddot{v}_n^- - \ddot{v}_n^+) &= -\frac{3c_{12}}{b}(u_n^- + u_{n-1}^+)_{,1} - \frac{3c_{22}}{b^2}(v_n - v_{n-1}^+) + \frac{3}{b}(\sigma_n^- + \sigma_{n-1}^+) + (\tau_n^- - \tau_{n-1}^+)_{,1} \\
\rho(\ddot{u}_n^- - \ddot{u}_{n-1}^+) &= \hat{c}_{11}(u_n^- - u_{n-1}^+)_{,11} - \frac{3c_{66}}{b^2}(u_n^- - u_{n-1}^+) - \frac{3c_{66}}{b}(v_n^- + v_{n-1}^+)_{,1} \\
&\quad + \frac{c_{12}}{c_{22}}(\sigma_n^- - \sigma_{n-1}^+)_{,1} + \frac{3}{b}(\tau_n^- + \tau_{n-1}^+) \\
\rho(\ddot{v}_n^- + \ddot{v}_{n-1}^+) &= c_{66}\{\frac{1}{b}(u_n^- - u_{n-1}^+)_{,1} + (v_n^- + v_{n-1}^+)_{,11}\} + \frac{1}{b}(\sigma_n^- - \sigma_{n-1}^+)
\end{aligned}
\tag{III-6}
$$

Viscoelastic Layer

Since the thickness of the elastomer is thin compared with the elastic layer, we can assume that the stress field is uniform through the thickness of the elastomer. In other words, we have $\sigma_n^- = \sigma_n^+ = \sigma_n$ and $\tau_n^- = \tau_n^+ = \tau_n$ for (III-6). Therefore, the following compatibility conditions for the

elastomer can be obtained immediately:

$$\varepsilon_{12(n)} = \frac{1}{4}\frac{\partial}{\partial x_1}(v_n^+ + v_n^-) + \frac{1}{2D}(u_n^+ - u_n^-)$$

$$\varepsilon_{22(n)} = \frac{1}{D}(v_n^+ - v_n^-) \tag{III-7}$$

where D is the thickness of the elastomer.

We further assume that the dissipation is mostly due to shear motion, i.e., that the normal component of the continuous traction vector is transmitted through the viscoelastic layer purely elastically. Therefore, by combining (III-7) with (III-3) we find

$$\sigma_n = \frac{E}{D}(v_n^+ - v_n^-)$$

$$\tau_n = \int_{-\infty}^{t} G(t-\tau)\{\frac{1}{2}\frac{\partial}{\partial x_1}(\dot{v}_n^+ + \dot{v}_n^-) + \frac{1}{D}(\dot{u}_n^+ - \dot{u}_n^-)\}d\tau \quad . \tag{III-8}$$

These two equations and four more from Eq. (III-6) are the complete equations needed to solve the impact on a composite plate with elastomer. For a plate made of N elastic layers and $(N-1)$ viscoelastic layers Eq. (III-6) provides $4N$ equations and (III-8) gives $2(N-1)$ equations. Since the total number of the unknowns are now $6N+2$ (u_o, v_o, σ_o, τ_o; u_1^-, v_1^-, u_1^+, v_1^+, σ_1, τ_1; ... ...; u_{N-1}^-, v_{N-1}^-, u_{N-1}^+, v_{N-1}^+, σ_{N-1}, τ_{N-1}; u_N, v_N, σ_N, τ_N) we can solve this system of equations with four additional conditions supplied by the suitable boundary conditions.

Here we notice that the governing equations are now a set of six difference-integro-partial differential equations. These equations can be

reduced to difference equations after appropriate integral transforms and the resulting difference equations can be handled rather simply, as in the previous chapter.

3. Numerical Results and Discussion

Impact on Plate

For the report we examine the impact on a plate consisting of two elastic layers and a viscoelastic layer, as shown in Fig. 17, with an impact function

$$\begin{aligned} \sigma_o &= P_o\{1-(\frac{x_1}{a})^2\} \sin\frac{\pi t}{t_o} \quad ; \quad |x_1| < a \quad \text{and} \quad 0 < t < t_o \\ &= 0 \quad ; \quad |x_1| > a \quad \text{or} \quad t > t_o, \text{or} \quad t < 0 \end{aligned} \qquad \text{(III-9)}$$

with all other stress components vanishing on both surfaces of the plate. Now by putting $n = 1$ and 2 into Eq. (III-6) we have eight equations and two more equations are obtained from Eq. (III-8). We again normalize these equations and take the integral transform, as in Chapter II. The resulting equations are:

$$
\begin{aligned}
&-(fs^2+\frac{C_{11}}{\Delta}bk^2)(\hat{\hat{U}}_1^-+\hat{\hat{U}}_o)-C_{12}ik(\hat{\hat{V}}_1^--\hat{\hat{V}}_o)+\hat{\hat{T}}_1 &&= 0 \\
&3C_{12}ik(\hat{\hat{U}}_1^-+\hat{\hat{U}}_o)-(fs^2+\frac{3C_{22}\Delta}{b})(\hat{\hat{V}}_1^--\hat{\hat{V}}_o)+3\hat{\hat{\Sigma}}_1-ik\frac{b}{\Delta}\hat{\hat{T}}_1 &&= -3\hat{\hat{\Sigma}}_o \\
&-(fs^2+\frac{b}{\Delta}C_{66}k^2)(\hat{\hat{V}}_1^-+\hat{\hat{V}}_o)-C_{66}ik(\hat{\hat{U}}_1^--\hat{\hat{U}}_o)+\hat{\hat{\Sigma}}_1 &&= \hat{\hat{\Sigma}}_o \\
&-(fs^2+\frac{b}{\Delta}\hat{C}_{11}k^2+\frac{\Delta}{b}3C_{66})(\hat{\hat{U}}_1^--\hat{\hat{U}}_o)+3C_{66}ik(\hat{\hat{V}}_1^-+\hat{\hat{V}}_o)-\frac{C_{12}}{C_{22}}\frac{b}{\Delta}ik\hat{\hat{\Sigma}}_1+3\hat{\hat{T}}_1 &&= -\frac{C_{12}}{C_{22}}\frac{b}{\Delta}ik\hat{\hat{\Sigma}}_o \\
&-(fs^2+\frac{C_{11}}{\Delta}bk^2)(\hat{\hat{U}}_2+\hat{\hat{U}}_1^+)-C_{12}ik(\hat{\hat{V}}_2-\hat{\hat{V}}_1^+)-\hat{\hat{T}}_1 &&= 0 \\
&3C_{12}ik(\hat{\hat{U}}_2+\hat{\hat{U}}_1^+)-(fs^2+\frac{3C_{22}\Delta}{b})(\hat{\hat{V}}_2-\hat{\hat{V}}_1^+)+3\hat{\hat{\Sigma}}_1+\frac{b}{\Delta}ik\hat{\hat{T}}_1 &&= 0 \\
&-(fs^2+\frac{b}{\Delta}C_{66}k^2)(\hat{\hat{V}}_2+\hat{\hat{V}}_1^+)-C_{66}ik(\hat{\hat{U}}_2-\hat{\hat{U}}_1^+)-\hat{\hat{\Sigma}}_1 &&= 0 \\
&-(fs^2+\frac{b}{\Delta}\hat{C}_{11}k^2+\frac{\Delta}{b}3C_{66})(\hat{\hat{U}}_2-\hat{\hat{U}}_1^+)+3C_{66}ik(\hat{\hat{V}}_2+\hat{\hat{V}}_1^+)+\frac{C_{12}}{C_{22}}\frac{b}{\Delta}ik\hat{\hat{\Sigma}}_1+3\hat{\hat{T}}_1 &&= 0 \\
&\hat{\hat{\Sigma}}_1 = E\frac{\Delta}{D}(\hat{\hat{V}}_1^+-\hat{\hat{V}}_1^-) \\
&\hat{\hat{T}}_1 = \bar{G}(s)\{\frac{\Delta}{D}(\hat{\hat{U}}_1^+-\hat{\hat{U}}_1^-)-\frac{ik}{2}(\hat{\hat{V}}_1^++\hat{\hat{V}}_1^-)\}
\end{aligned}
\tag{III-10}
$$

where $\bar{G}(s)$ is the Laplace transform of the relaxation function $G(t)$ with respect to $\tau = t/T_o$ and we have used the boundary conditions $\tau_o = \tau_2 = \sigma_2 = 0$. From the above 10 equations we can find 10 unknowns $(\hat{\hat{U}}_o, \hat{\hat{V}}_o, \hat{\hat{U}}_1^-, \hat{\hat{V}}_1^-, \hat{\hat{U}}_1^+, \hat{\hat{V}}_1^+, \hat{\hat{\Sigma}}_1, \hat{\hat{T}}_1; \hat{\hat{U}}_2, \hat{\hat{V}}_2)$ with given impact function $\hat{\hat{\Sigma}}_o$ and once these are calculated the displacement and the stress fields can be computed by inversions of the integral transforms by means of the FFT alogorithm.

Numerical Results

For the present computation we have used the Young's modulus $E = .7*10^4$ psi and the shear modulus $G'(\omega) = .817*19^4 - \frac{2.41*10^{12}}{3*10^4+\omega^2}$ for the elastomer where ω is given in hertz. The $G'(\omega)$ in this case implies that $G'(\infty) = .817*10^4$ psi and $\max(G''(\omega)/G'(\omega)) = 3.3$ at $\omega_o = 800$ Hz.

The propagation of stress wave in this case is quite similar to that of the composite plate without an elastomer layer except the peak values of the interlaminar stress. Values of the peak stress with different thickness of the elastomer layer are plotted with those of the purely elastic plate in Fig. 18. As we can see in this figure the interlaminar shear stress has increased by a small amount while a reduction of the normal stress is considerable when the elastomer layer becomes thicker and thicker. From this result it is obvious that the reduction of the normal component of stress can be achieved by introducing such a soft and energy-dissipating elastomer layer.

Discussion

In addition to the simple reduction of the normal stress it is also observed that the amount of reduction increases with the value of $G''(\omega)/G'(\omega)$ and the location of ω_o at which $G''(\omega)/G'(\omega)$ becomes the maximum value. In other words, we can make the dissipation effect more serious by choosing an elastomer whose $G''(\omega)/G'(\omega)$ becomes maximum at ω_o around which the most of the impact energy is carried out.

It is also believed that a further dissipation effect will be possible if we make the transmission of the normal stress viscoelastic across the elastomer layer, which we have assumed is elastic for this report.

IV. IMPACT ON A PLATE WITH A CRACK

1. Introduction

When the impact stress is low, the impact is elastic and the stresses in the plate can be described by elastic wave propagation. When the stress is increased beyond a certain limit then the impact damage occurs. Elastic-plastic impact is complicated for two reasons, namely, unloading and loading must be treated differently, and the strain rate effect [30] must be included. If the impact stress is increased further to a certain level where the induced stress is higher than the strength of a target material then penetration begins to occur. In this limit the target material sometimes behaves as a fluid and such a state of impact is known as a hydraulic impact [31]. Another failure mode is the occurrence of interlaminar cracks.

Investigation of the stress state in solids with cracks falls in the category of so-called fracture mechanics and has been under an extensive scrutiny since the famous enunciation by Griffith [32]. Presence of cracks inside a material usually leads us to a mixed boundary value problem and only a limited class of problems can be solved [33,34]. In the case of dynamic loading the problem becomes more difficult due to the scattering of the stress wave by the crack [21-24]. In this report we will formulate the problem of a plate with a crack which is subject to a dynamic loading.

Our original goal was to study the effect of interlaminar cracks in composite plates in response to impact loads. Debugging problems in other parts of this report,however,used valuable time originally set aside for this problem. The following section is an attempt to illustrate the

use of the Mindlin plate theory for the study of interlaminar cracks and to point out the mathematical difficulties that must be overcome in solving the problem.

2. Formulation

Description of Problem

The plate under consideration has a crack on the midplane running from $x_1 = -h$ to $+h$ as shown in Fig. 19. Stress can be applied either on the surface of the plate or on the crack surface. In the former case the crack surfaces can be in contact and the boundary conditions become more complex due to the partial continuity of stresses and displacements during the contact. For the present report to illustrate the mathematical difficulties we assume that the crack surface is subject to a known compressive impact.

Governing Equation and Boundary Conditions

We can formulate this crack problem by assuming that the lower and the upper half plates are made of a number of layers but for simplicity we consider the plate to consist of two identical layers and the crack to be present on the interface of these two layers. Following the notation shown in Fig. 19 we have the governing equations identical to Eq. (III-6) with $n = 1$ and 2. The boundary condition requires that both plate surfaces remain traction free. The crack surface is subject to a prescribed impact condition while the displacement and stress are continuous along the layer boundary outside the crack. Namely, we have

$$
\begin{aligned}
&\sigma_o = \tau_o = \sigma_2 = \tau_2 = 0 \\
&\left.\begin{aligned} \sigma_1^+ &= \sigma_1^- = -P_o(x_1,t) \\ \tau_1^+ &= \tau_1^- = 0 \end{aligned}\right\} |x| < h \\
&\left.\begin{aligned} u_1^+ &= u_1^- , \quad v_1^+ = v_1^- \\ \sigma_1^+ &= \sigma_1^- , \quad \tau_1^+ = \tau_1^- \end{aligned}\right\} |x| > h \ .
\end{aligned}
\qquad \text{(IV-1)}
$$

Due to the twofold symmetry of the problem we now have $u_o = u_2$, $v_o = -v_2$, $\tau_1^+ = \tau_1^- = 0$ and we can set $u_1^+ = u_1^- = u_1$, $-v_1^+ = v_1^- = v_1$. Thus, the eight equations obtained from Eq. (III-6) are now reduced to

$$\begin{aligned}
\rho(\ddot{u}_1+\ddot{u}_o) &= c_{11}(u_1+u_o)_{,11} + \frac{c_{12}}{b}(v_1-v_o)_{,1} \\
\rho(\ddot{v}_1-\ddot{v}_o) &= -\frac{3c_{12}}{b}(u_1+u_o)_{,1} - \frac{3c_{22}}{b^2}(v_1-v_o) + \frac{3\sigma}{b} \\
\rho(\ddot{u}_1-\ddot{u}_o) &= \hat{c}_{11}(u_1-u_o)_{,11} - \frac{3c_{66}}{b^2}(u_1-u_o) - \frac{3c_{66}}{b}(v_1+v_o)_{,1} + \frac{c_{12}}{c_{22}}\sigma_{,1} \\
\rho(\ddot{v}_1+\ddot{v}_o) &= c_{66}\{\frac{1}{b}(u_1-u_o)_{,1} + (v_1+v_o)_{,11}\} + \frac{\sigma}{b}
\end{aligned} \tag{IV-2}$$

and the boundary condition is now

$$\left.\begin{aligned}
\sigma &= -P_o(x_1,t) \qquad |x_1| < h \\
v_1 &= 0 \qquad\qquad\quad\; |x_1| > h
\end{aligned}\right\} \quad \text{along} \quad x_2 = 0 \tag{IV-3}$$

Dual Integral Equation

We now normalize the governing equation (IV-2) and take the integral transform. Then we have

$$\begin{bmatrix}
(fs^2+\frac{b}{\Delta}C_{11}k^2), & C_{12}ik, & 0, & 0 \\
-3C_{12}ik, & (fs^2+\frac{3\Delta}{b}C_{22}), & 0, & 0 \\
0, & 0, & C_{66}ik, & (fs^2+\frac{b}{\Delta}C_{66}k^2) \\
0, & 0, & (fs^2+\frac{b}{\Delta}\hat{C}_{11}k^2+\frac{3\Delta}{b}C_{66}), & -3C_{66}ik
\end{bmatrix}
\cdot
\begin{bmatrix}
\hat{\bar{U}}_1+\hat{\bar{U}}_o \\
\hat{\bar{V}}_1-\hat{\bar{V}}_o \\
\hat{\bar{U}}_1-\hat{\bar{U}}_o \\
\hat{\bar{V}}_1+\hat{\bar{V}}_o
\end{bmatrix}
=
\begin{bmatrix}
0 \\
3\hat{\bar{\Sigma}} \\
\hat{\bar{\Sigma}} \\
-\frac{C_{12}}{C_{22}}\frac{b}{\Delta}ik\hat{\bar{\Sigma}}
\end{bmatrix} \tag{IV-4}$$

and these can be solved for $\hat{\hat{U}}_o$, $\hat{\hat{U}}_1$, $\hat{\hat{V}}_o$, and $\hat{\hat{V}}_1$ in terms of $\hat{\hat{\Sigma}}$. Since the mixed boundary conditions are given by σ and v_1 we solve $\hat{\hat{V}}_1$ as

$$\hat{\hat{V}}_1 = K(s,k)\hat{\hat{\Sigma}} \tag{IV-5}$$

with

$$K(s,k) = \frac{1}{2}\left[\frac{3}{A}\left(fs^2 + \frac{b}{\Delta}C_{11}k^2\right) + \frac{1}{B}\left\{\frac{C_{12}}{C_{22}}\frac{b}{\Delta}C_{66}k^2 - \left(fs^2 + \frac{b}{\Delta}\hat{C}_{11}k^2 + \frac{3b}{\Delta}C_{66}\right)\right\}\right]$$

$$A = \mathrm{Det}\begin{vmatrix} \left(fs^2 + \frac{b}{\Delta}C_{11}k^2\right) & , \; C_{12}ik \\ -3C_{12}ik & , \; \left(fs^2 + \frac{3\Delta}{b}C_{22}\right) \end{vmatrix} \tag{IV-6}$$

$$B = \mathrm{Det}\begin{vmatrix} C_{66}ik & , \; \left(fs^2 + \frac{b}{\Delta}C_{66}k^2\right) \\ \left(fs^2 + \frac{b}{\Delta}\hat{C}_{11}k^2 + \frac{3\Delta}{b}C_{66}\right) & , \; -3C_{66}ik \end{vmatrix} .$$

Next we take the inverse transform of $\hat{\hat{\Sigma}}$ and $\hat{\hat{V}}_1$, and apply the mixed boundary condition given in Eq. (IV-3). Since the boundary conditions are for all times $t > 0$ we only take the inverse Fourier transform to apply the boundary conditions, i.e.,

$$\begin{aligned} \hat{\Sigma}(\eta,s) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{\hat{\Sigma}}\, e^{-ik\eta}dk \\ \hat{V}_1(\eta,s) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} K(s,k)\,\hat{\hat{\Sigma}}\, e^{-ik\eta}dk \quad . \end{aligned} \tag{IV-7}$$

Application of the boundary condition given by Eq. (IV-3) results in the following integral equation:

$$-\hat{P}_o(\eta,s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\hat{\Sigma}}\, e^{-ik\eta} dk \quad : \quad |\eta| < h/\Delta$$

$$0 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} K(s,k)\, \hat{\hat{\Sigma}}\, e^{-ik\eta} dk \quad : \quad |\eta| > h/\Delta \qquad \text{(IV-8)}$$

for an unknown function $\hat{\hat{\Sigma}}$.

3. Discussion

The integral equations of the type given in Eq. (IV-8) are known as dual integral equations, each of which has its own region of application and occur in mixed boundary value problems [35]. There are a number of ways to solve this type of integral equations, such as by reduction to a single Fredholm integral equation or by using the Wiener-Hopf technique [36]. Finding the solution depends on the kernel and in general it is rather difficult to do except for some special cases such as for Bessel kernels or trigonometric kernels.

Once the unknown function $\hat{\hat{\Sigma}}$ is determined the other variables $(\hat{\hat{U}}_o, \hat{\hat{U}}_1, \hat{\hat{V}}_o, \hat{\hat{V}}_1)$ can be computed by solving the algebraic equation (IV-4) and the complete displacement can be found by inversions of the integral transform.

The problem formulated in this chapter is the simplest impact problem in that the contact of the crack surface does not occur and that it has a twofold symmetry. But it is expected that the critical response of the plate, particularly the stress field near the crack, can be a guideline for a more complex problem.

V. CONCLUSION AND RECOMMENDED RESEARCH

The present theory is a generalization of Mindlin's approximate theory of plate applied to a multilayer plate under impact. By combined use of the finite difference technique in the thickness direction and integral transforms this model has been shown to be very effective for wave propagation analyses.

This model is extended to examine the effects of an elastomer layer between elastic layers of the plate. The reduction of interlaminar normal stress is signficant due to the damping layer but further investigation seems necessary to determine the nature of the reduction.

The presence of a crack in the plate has been formulated. The resulting equations are given by dual integral equations which, as in many cases, are rather difficult to solve.

The basic idea of the periodic structure of the multilayer plate, where the governing equations are derived for each layer and given by a set of difference-differential equations, may be useful to handle different types of problems,such as heat conduction and thermoelastic problems in composite plates.

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Figures

1. Composite Plate and Layer

2 a,b. Dispersion Relationship and Phase Velocity for Isotropic Plate: One-layer Model ($\lambda = \mu$).

3 a,b. Dispersion Relationship and Phase Velocity for Composite Plate: One-layer Model (55% Graphite Fiber-Epoxy Matrix, Layup Angle 45°).

4 a,b. Dispersion Relationship and Phase Velocity for Isotropic Plate: Two-layer Model ($\lambda = \mu$).

5 a,b. Dispersion Relationship and Phase Velocity for Composite Plate: Two-layer Model (55% Graphite Fiber-Epoxy Matrix, Layup Angle 45°).

6 a~f. Longitudinal Propagation Impact Stress in Isotropic Plate: Two-layer Model ($\lambda = \mu = 1.2*10^7$ psi; $\Delta = 1$ cm, $t_o = 10$ μsec, $a = 4$ cm).

7. Transverse Propagation of Normal Stress in Isotropic Plate: 4-layer Model ($\lambda = \mu = 1.2*10^7$ psi; $\Delta = 4$ cm, $t_o = 2$ μsec, $a = 40$ cm).

8. Transverse Propagation of Normal and Shear Stress in Isotropic Plate: (Same as in Fig. 7).

9. Transverse Propagation of Normal Stress in Composite Plate: 8-layer Model (55% Graphite Epoxy-Fiber Matrix, Layup Angle 15°; $\Delta = 1$ cm, $t_o = 2$ μsec, $a = 2$ cm).

10. Two Dimensional Propagation of Normal Stress in Isotropic Plate: 4-layer Model ($\lambda = \mu = 1.2*10^7$ psi; $\Delta = 4$ cm, $t_o = 4$ μsec, $a = 4$ cm).

11. Velocity Surface and Wave Surface of Composite Plate (55% Graphite Fiber-Epoxy Matrix, Layup Angle 45°).

12a,b. Comparison of Theoretical Wave Front and Numerical Wave Front of Composite Plate 10 μsec after Impact: 55% Graphite Fiber-Epoxy Matrix (For Numerical Results; 8-layer Model, $\Delta = 4$ cm, $t_o = 4$ μsec, $a = 2$ cm). Without and with Correction Factors.

13. Effect of Correction Factor or Transverse Propagation of Peak Value of Normal Stress: 8-layer Model (55% Graphite Fiber-Epoxy Matrix, Layup Angle 45°; $\Delta = 4$ cm, $t_o = 4$ μsec, $a = 2$ cm).

14. Viscoelastic Impact Energy Absorbing Models.

15a,b. Complex Modulus of Elastomer.

16. Plate with Viscoelastic Layers.

17. Impact of Plate Made of 2 Elastic Layers and a Viscoelastic Layer.

18. Peak Value of Interlaminar Stress Vs Elastomer thickness (Two Elastic Layers and a Viscoelastic Layer: (Δ = 1 cm, t_o = 10 μsec, a = 4 cm).

19. Composite Plate with Crack.

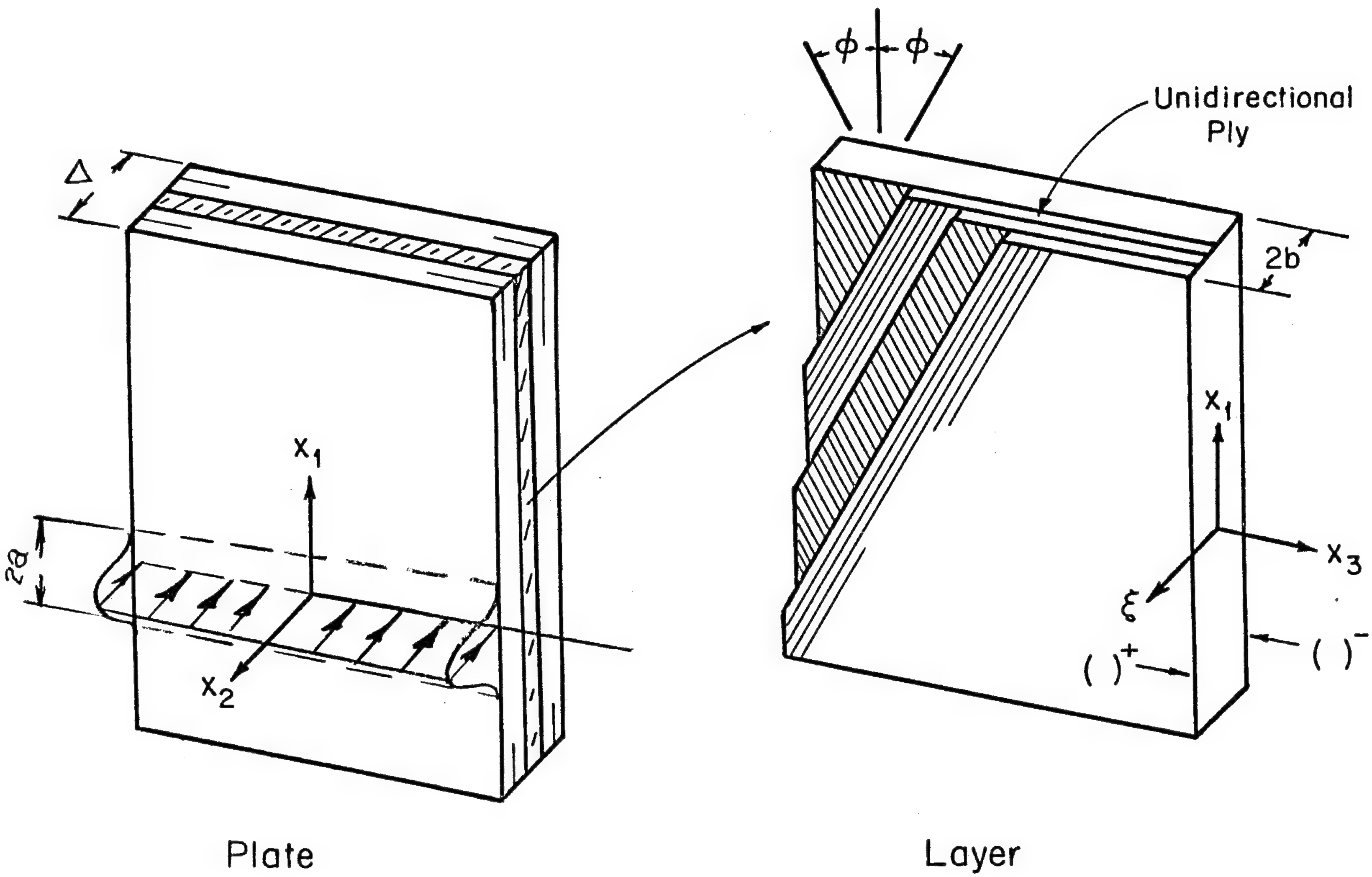


Figure 1. Multilayered Composite Plate and Layer

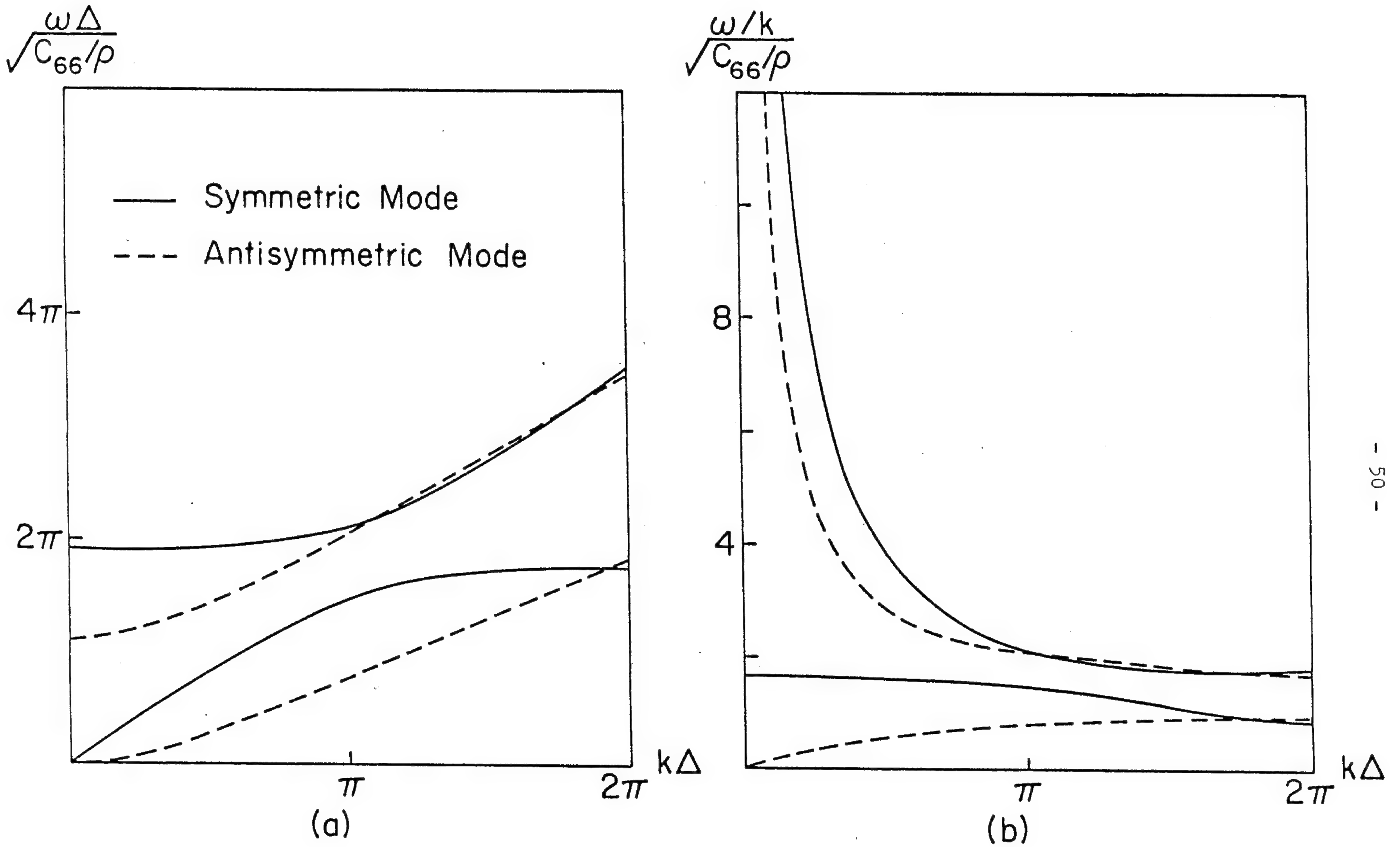


Figure 2. Dispersion Relationship and Phase Velocity of Isotropic Plate: One-Layer Model ($\lambda = \mu$, Poisson's Ration = 1/4)

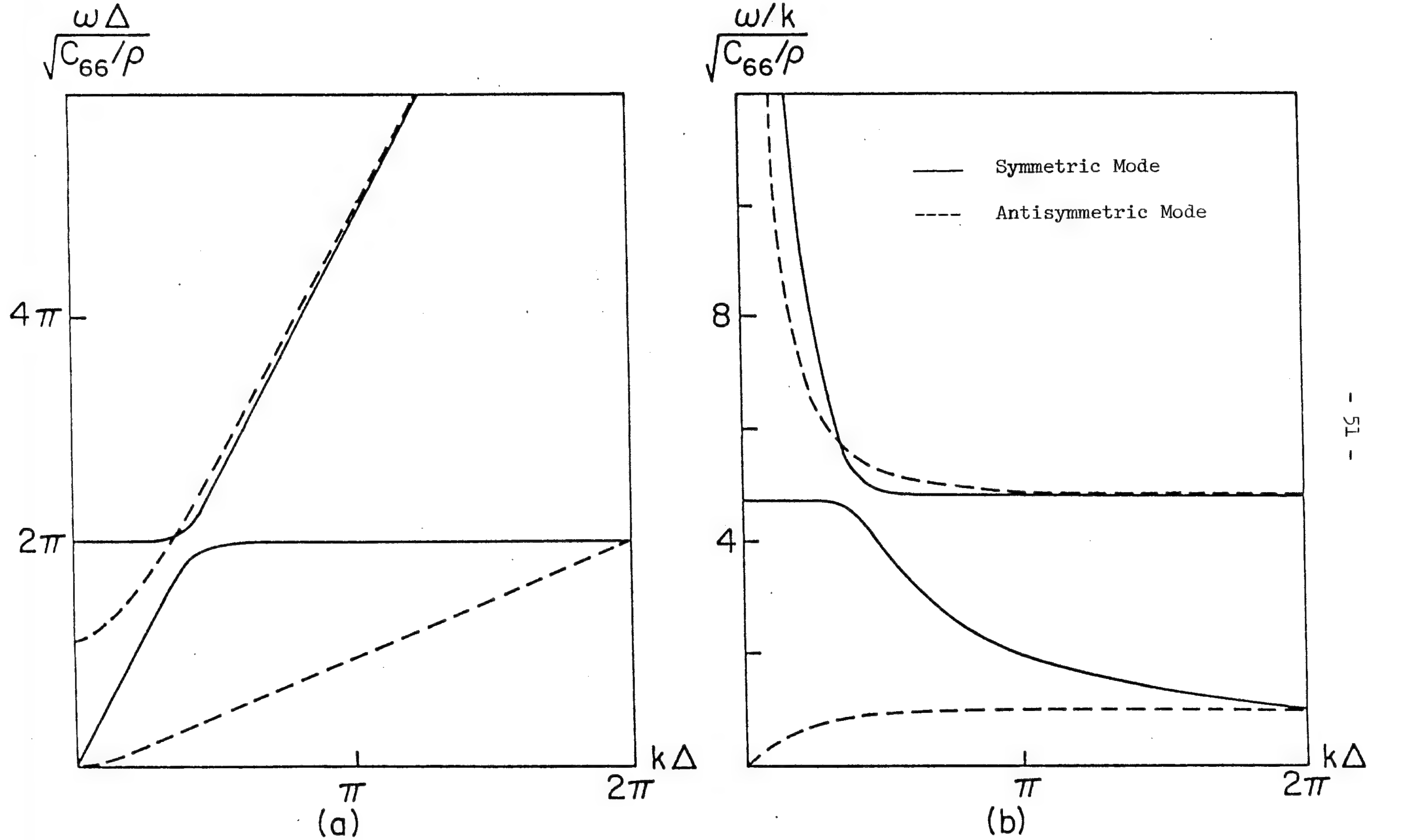


Figure 3. Dispersion Relationship and Phase Velocity of Composite Plate: One-Layer Model (55% Graphite Fiber - Epoxy Matrix; Layup Angle = 45°)

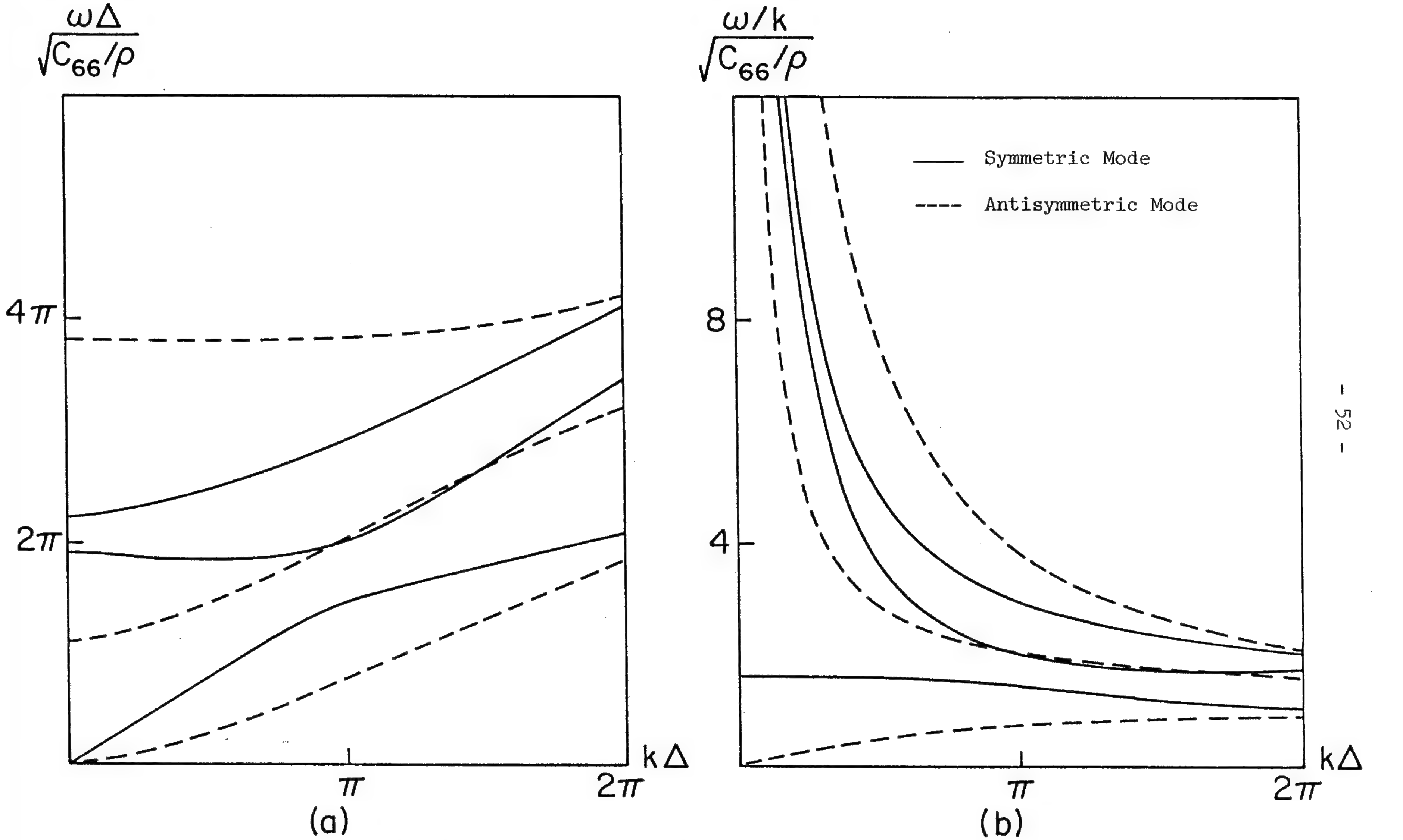


Figure 4. Dispersion Relationship and Phase Velocity of Isotropic Plate: Two-layer Model ($\lambda = \mu$, Poisson's Ratio = 1/4)

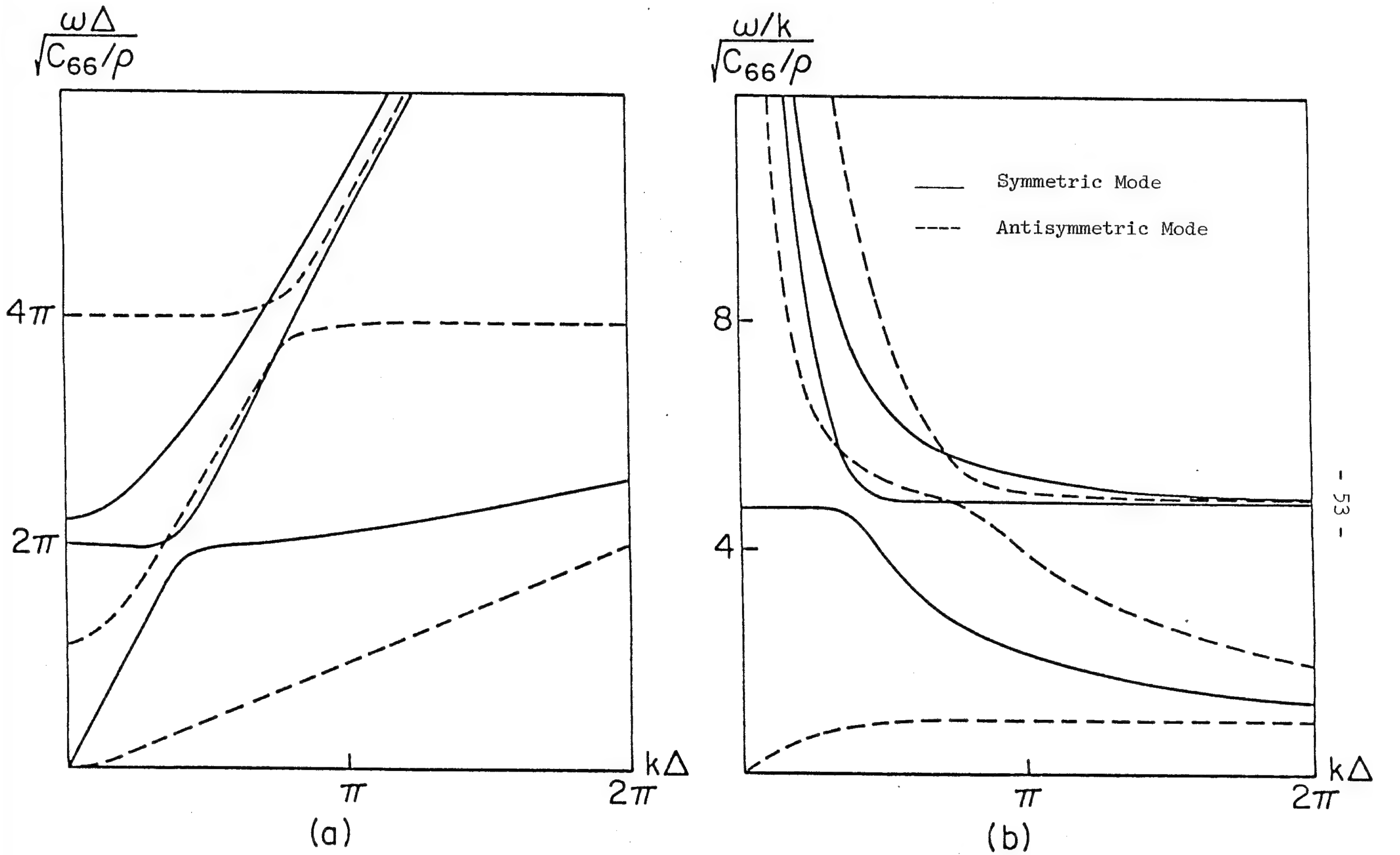


Figure 5. Dispersion Relationship and Phase Velocity of Composite Plate: Two-Layer Model (55% Graphite Fiber - Epoxy Matrix, Layup Angle = 45°)

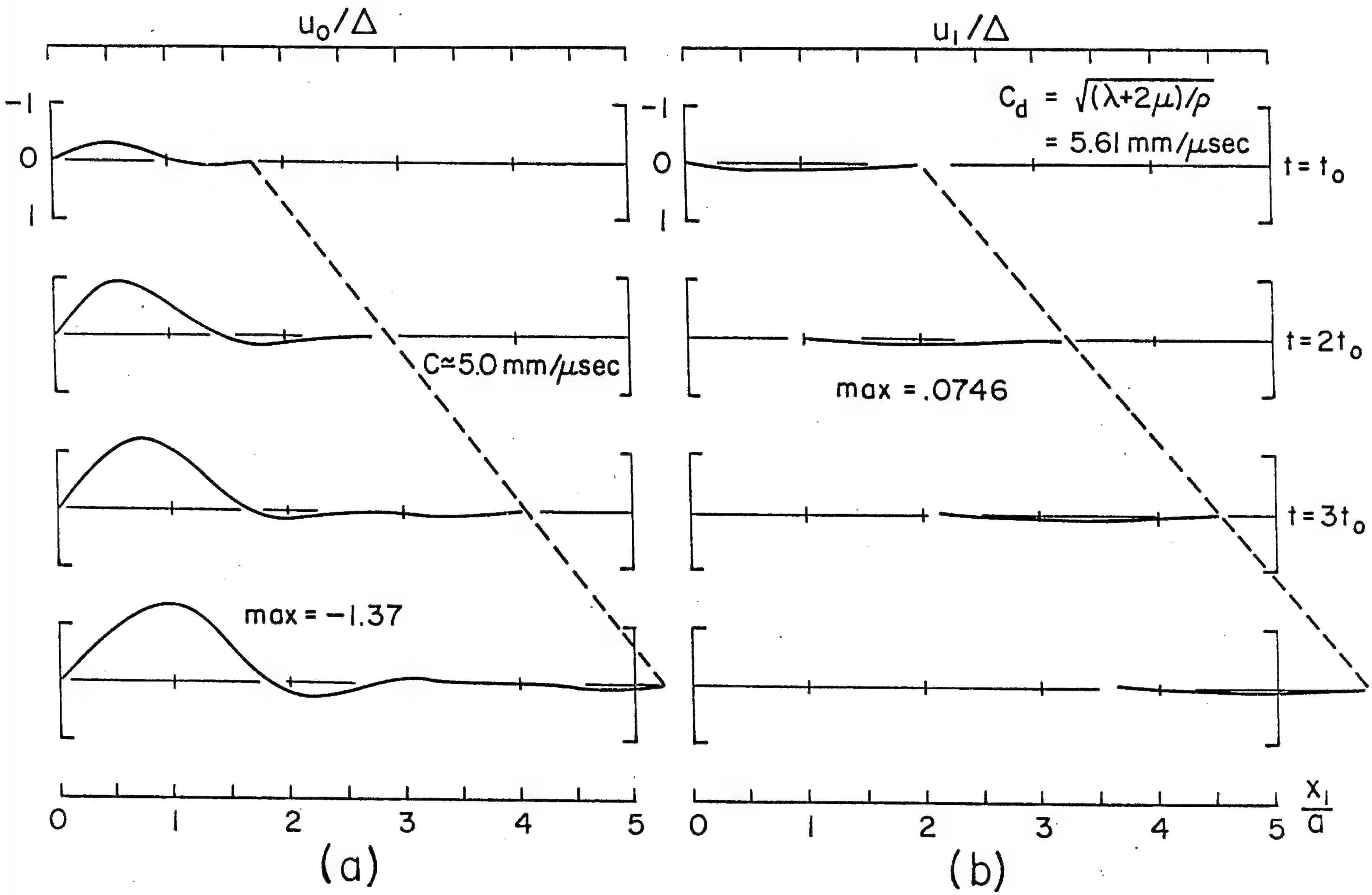


Figure 6 Longitudinal Propagation in Isotropic Plate; 2-layer Model
($\lambda = \mu = 1.2 \times 10^7$ psi; $\Delta = 1$ cm, $t_o = 10$ μsec, $a = 4$ cm)

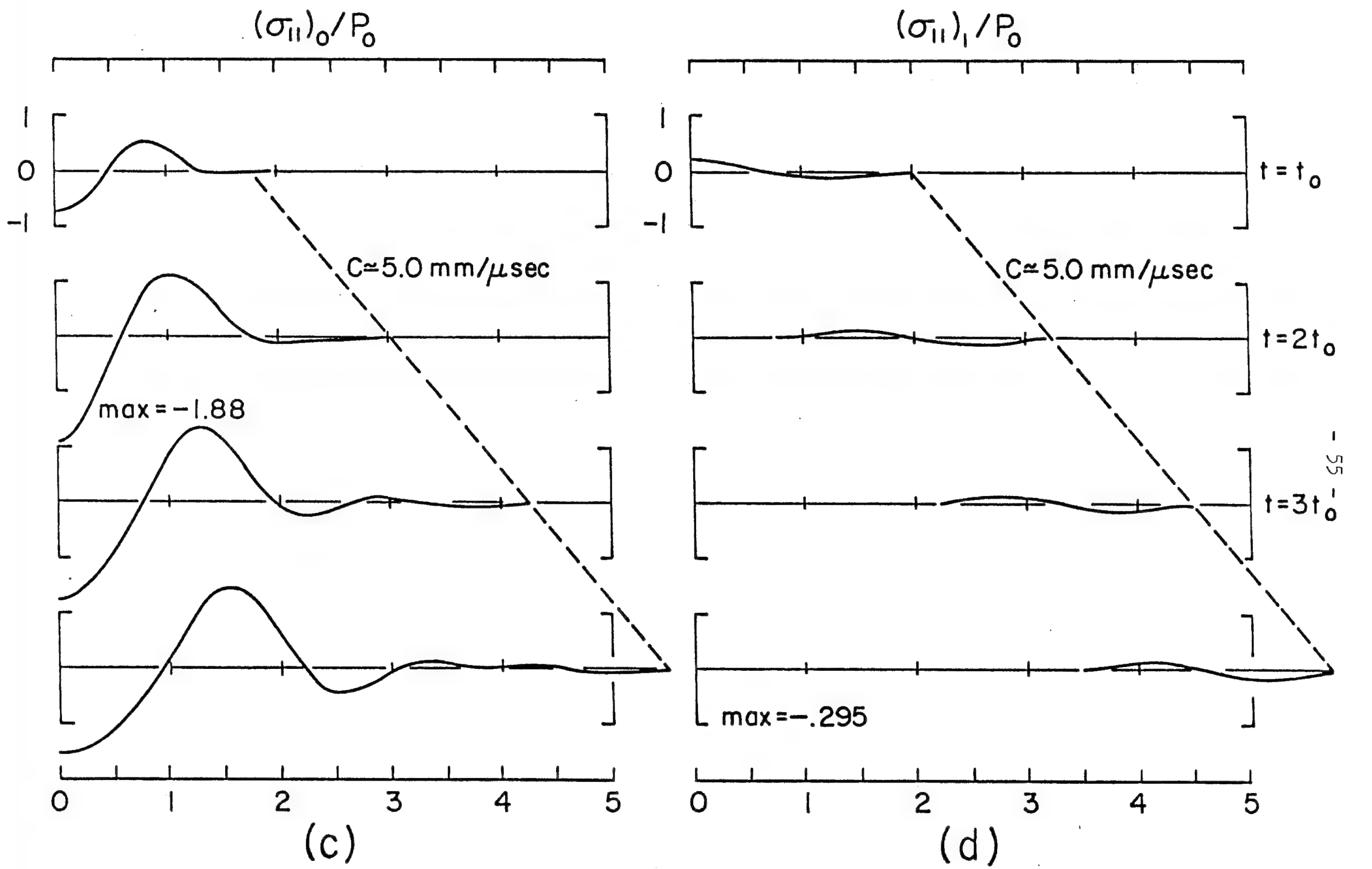


Figure 6 Continued

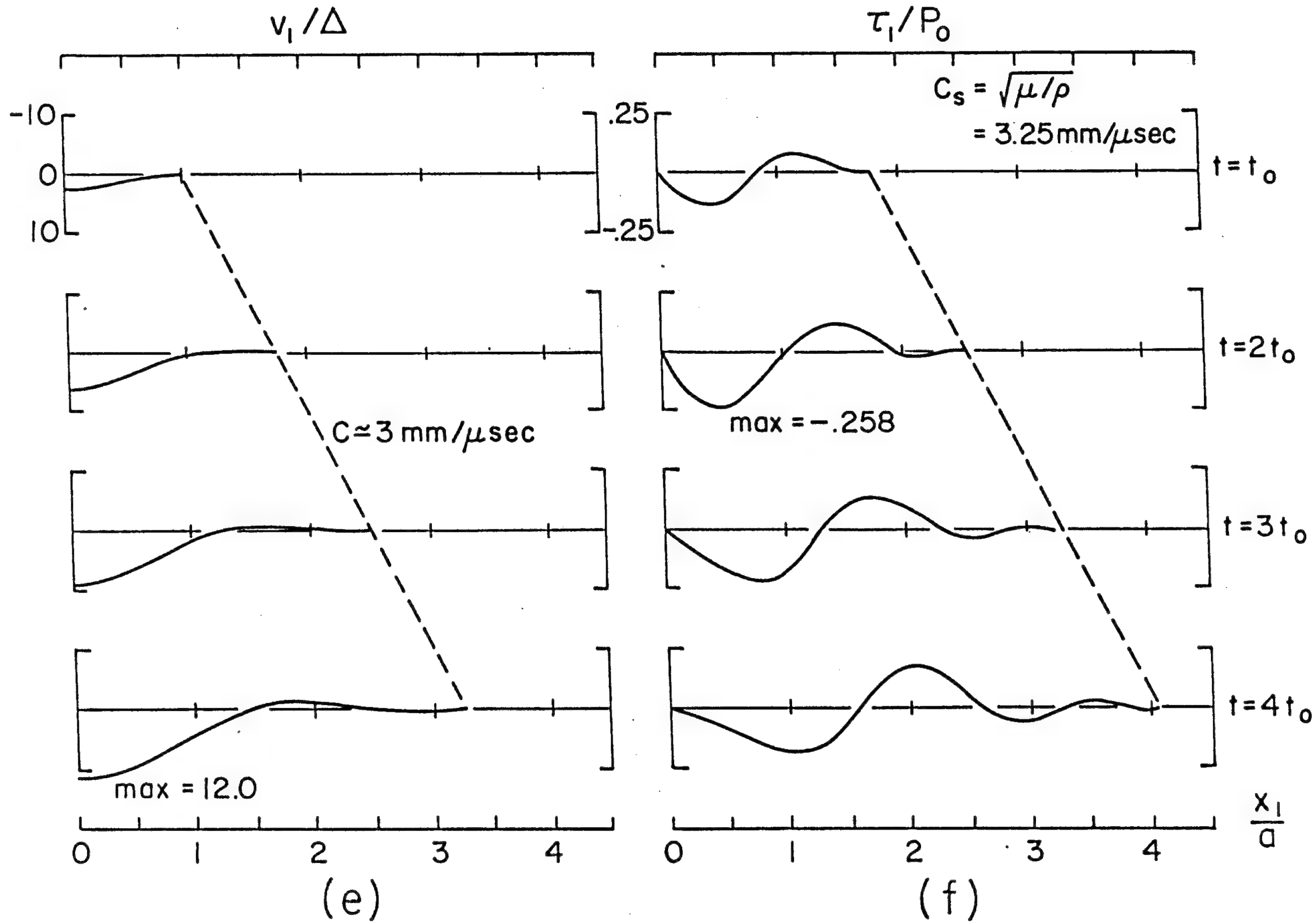


Figure 6 Continued

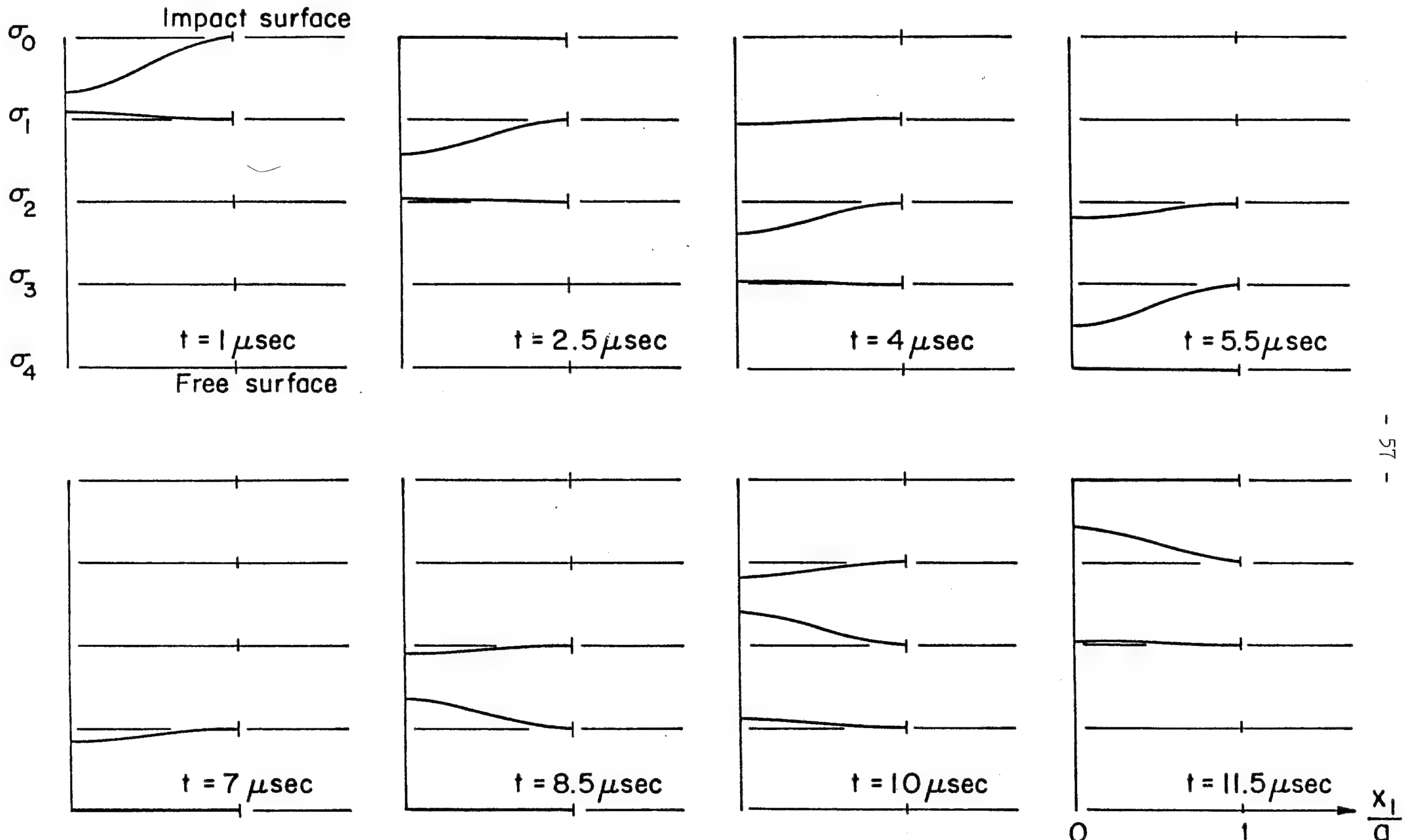


Figure 7. Transverse Propagation of Normal Stress in Isotropic Plate: 4-layer Model ($\lambda = \mu = 1.2*10^7$ psi; $\Delta = 4$ cm, $t_o = 2$ μsec, a = 40 cm)

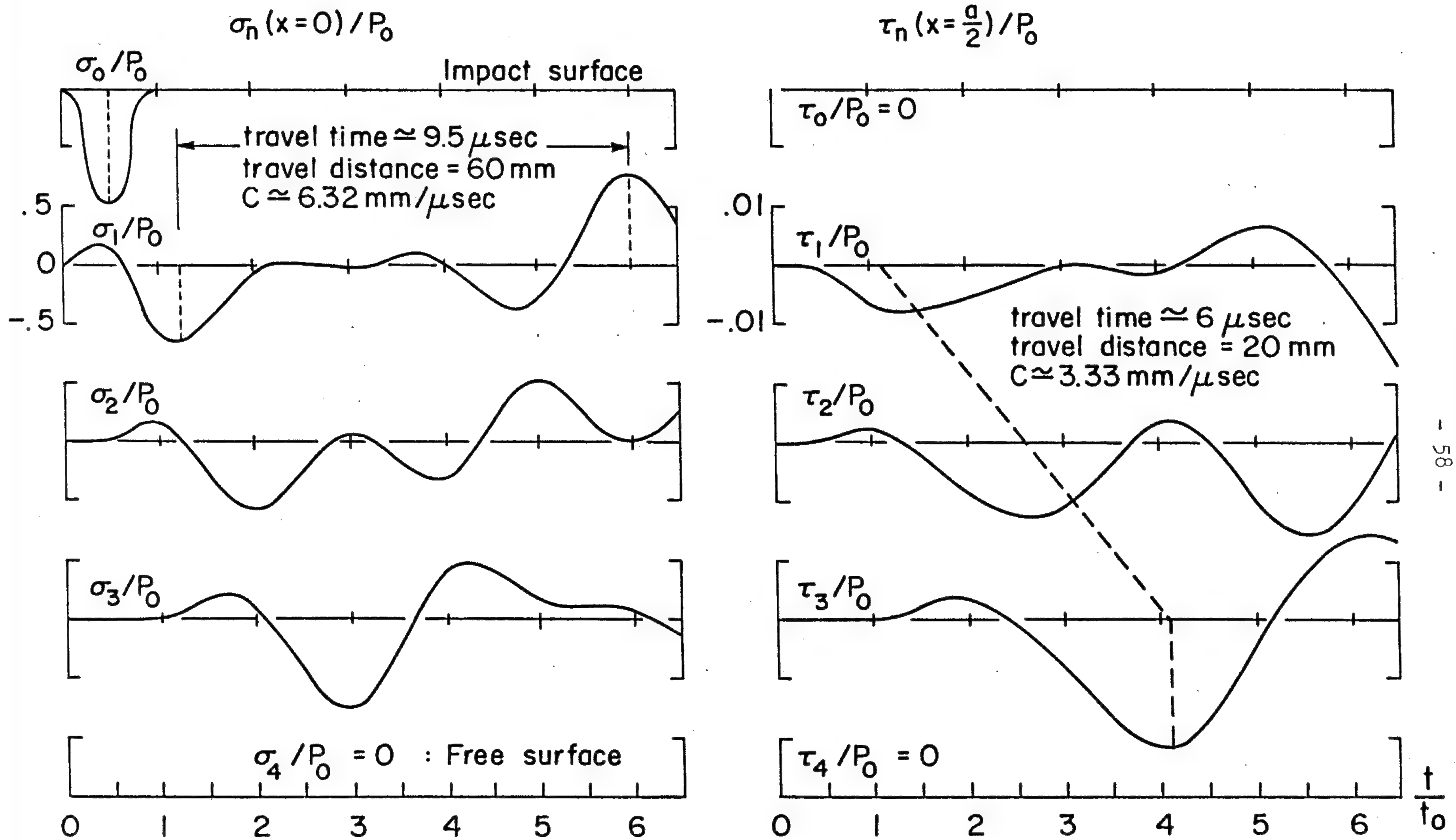


Figure 8. Transverse Propagation of Normal and Shear Stress (Same as in Figure 7)

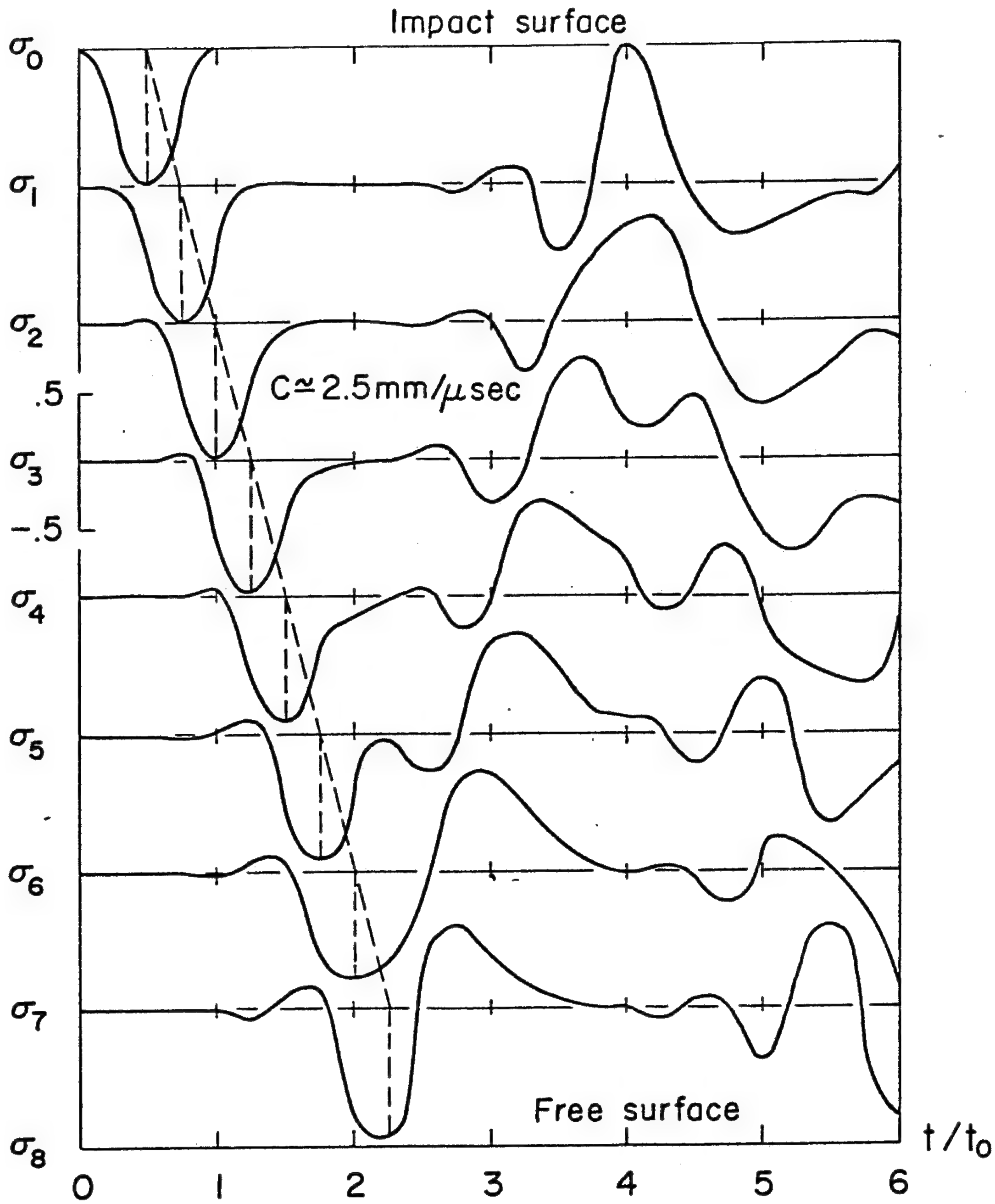


Figure 9 Transverse propagation of normal stress in composite plate; 8-layer Model (55% Graphite Fiber - Epoxy Matrix, ±15° Layup; Δ = 1 cm, t_o = 2 μsec, a = 2cm)

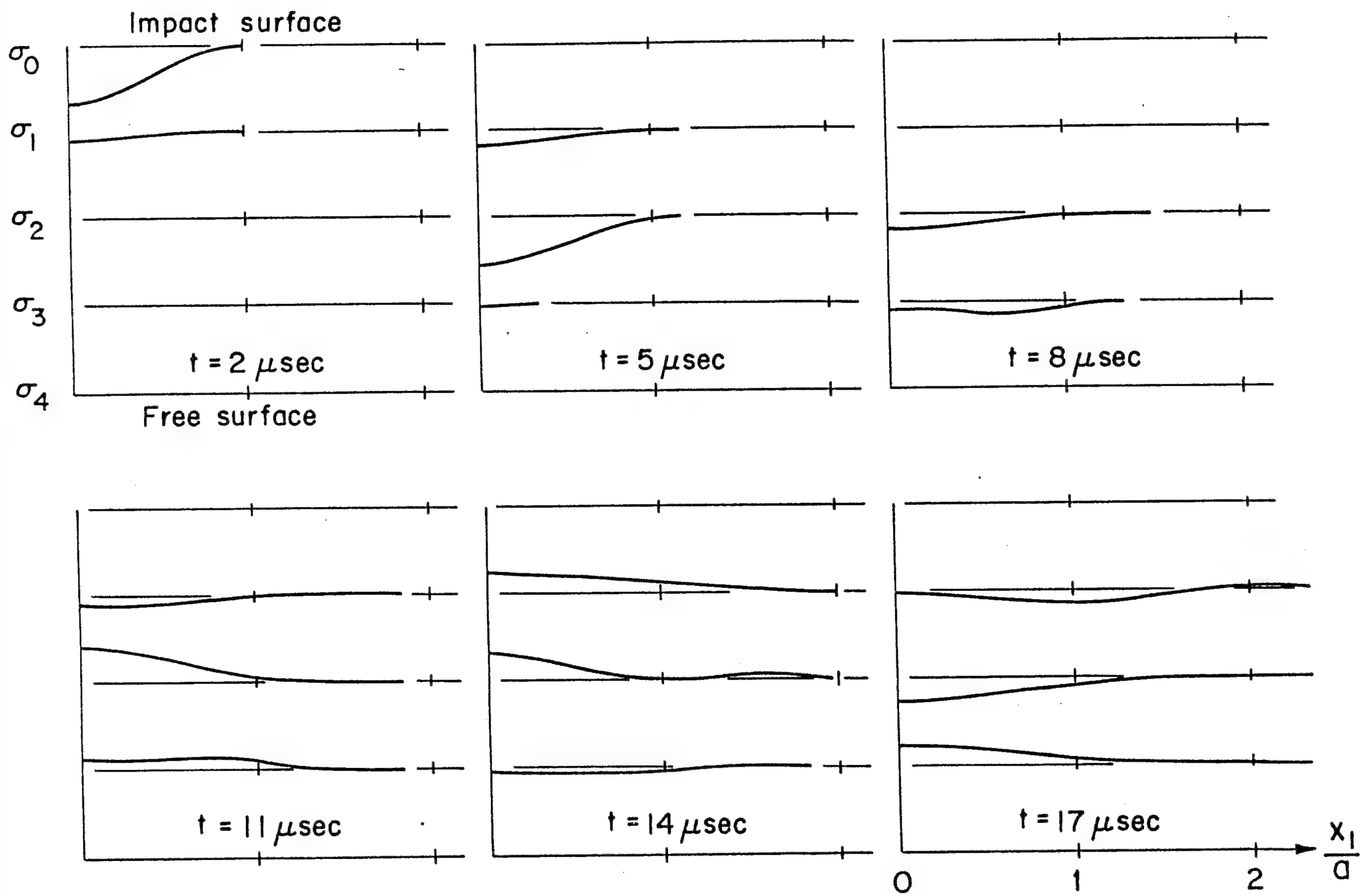


Figure 10. Two Dimensional Propagation of Normal Stress in Isotropic Plate: 4-layer Model ($\lambda = \mu = 1.2*10^7$ psi; $\Delta = 4$ cm, $t_o = 4$ μsec, $a = 4$ cm)

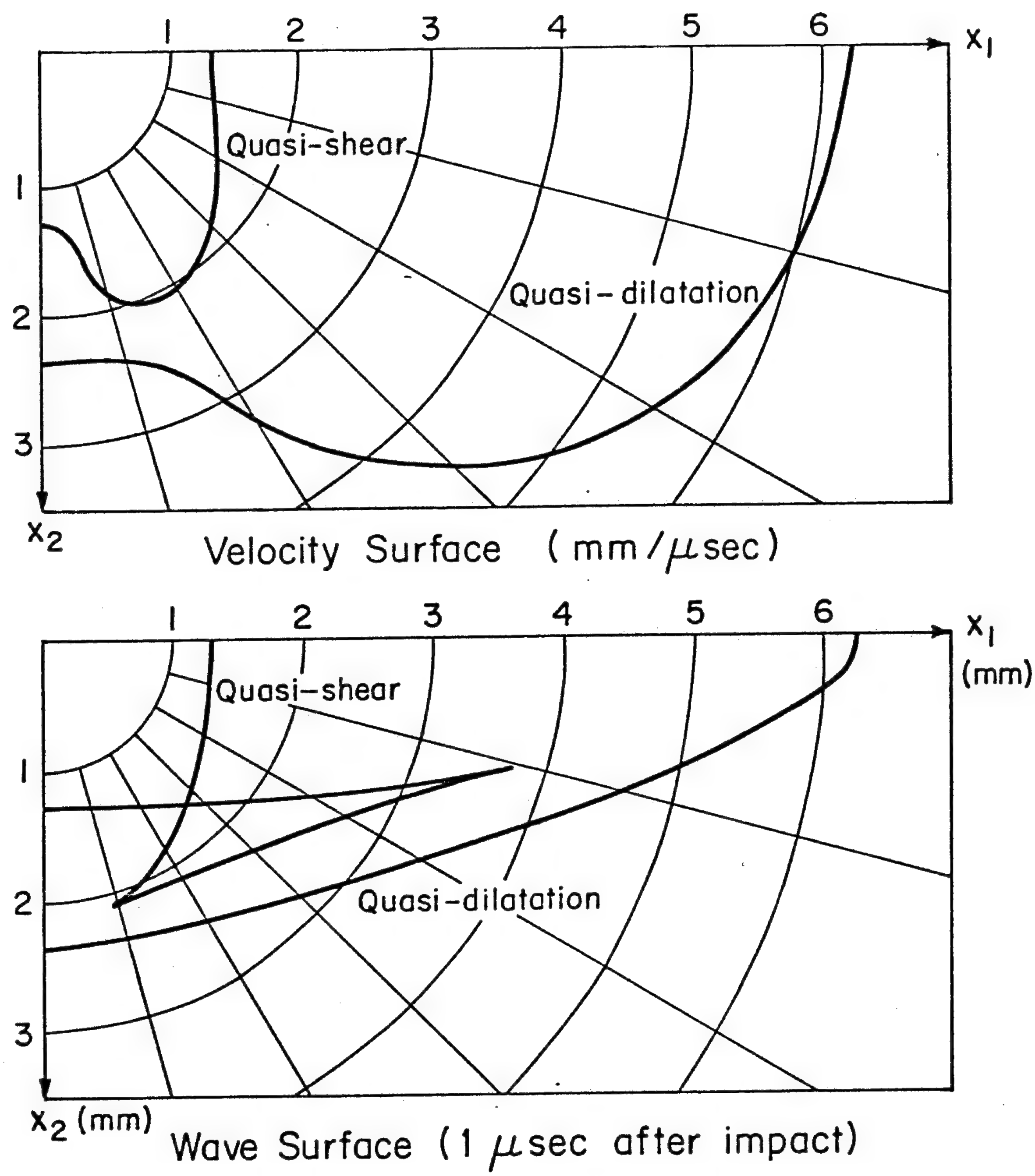


Figure 11. Velocity Surface and Wave Surface of Composite Plate (55% Graphite Fiber-Epoxy Matrix, Layup Angle 45°)

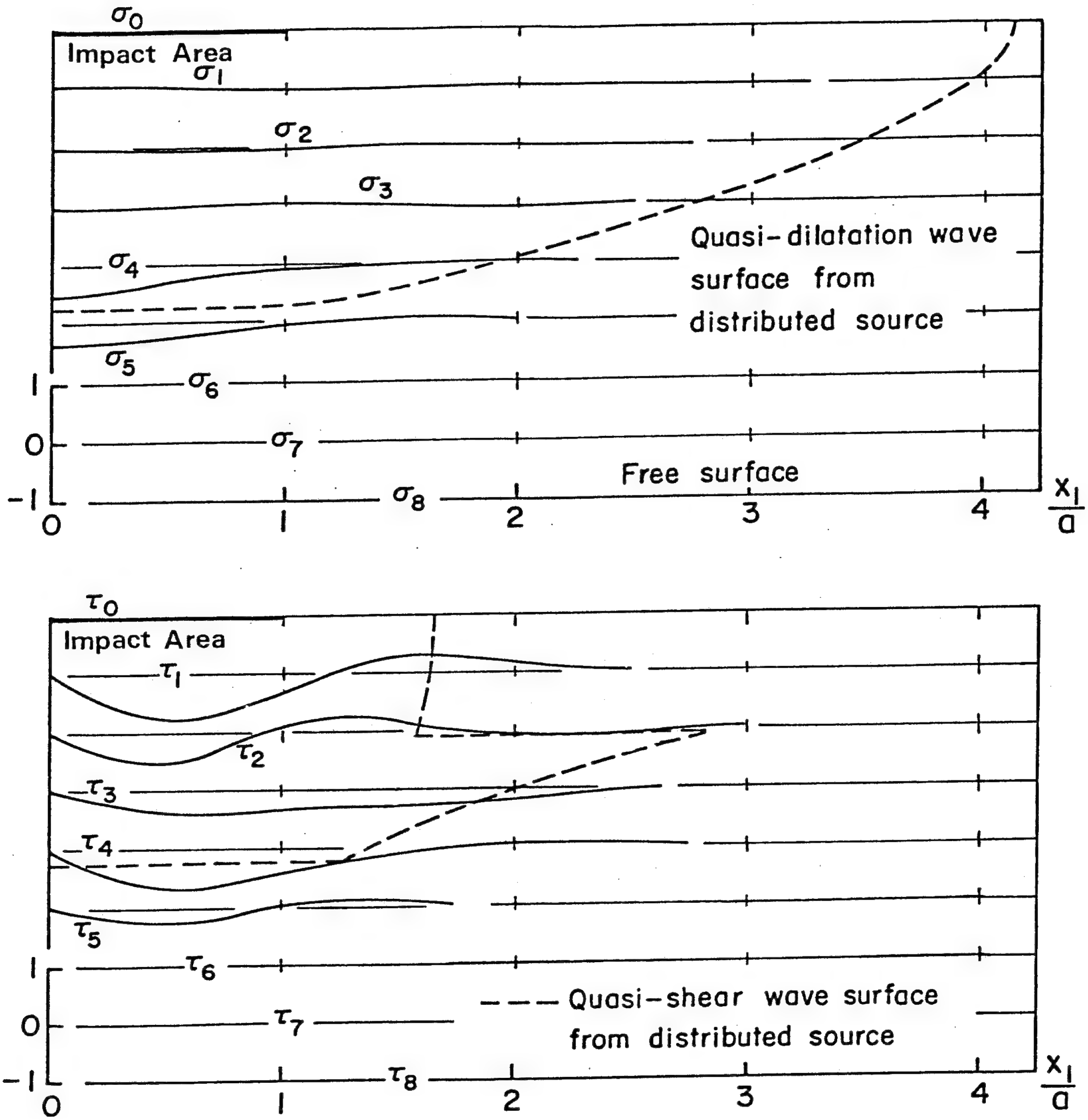


Figure 12a. Wave front 10 μsec after impact (without correction factor) (55% Graphite Fiber-Epoxy Matrix, $\pm 45^{\circ}$ Layup; Δ = 4 cm, 8-layer Model; t_o = 4 μsec, a = 2 cm)

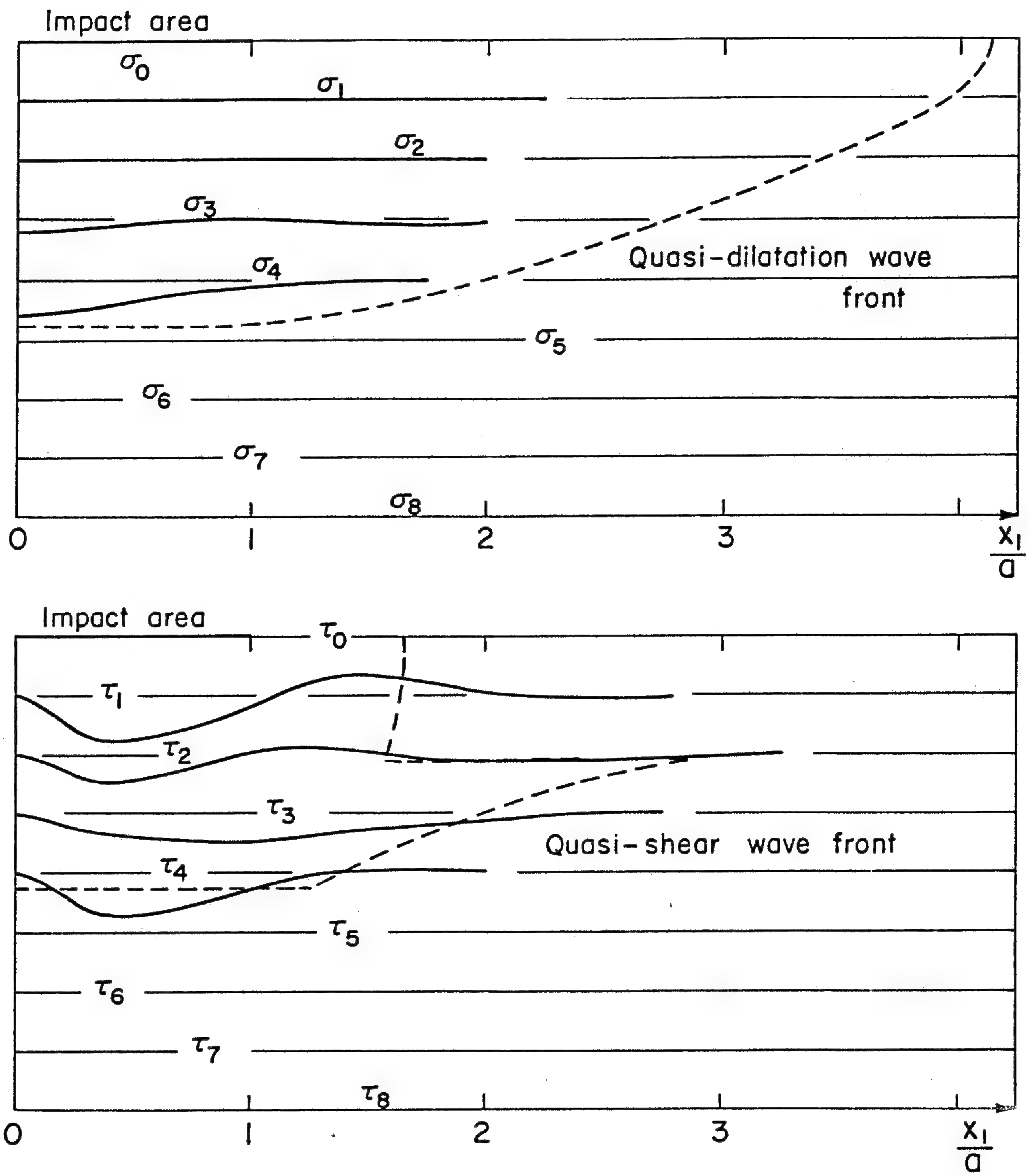


Figure 12b. Wave front 10 μsec after impact (with correction factor) (55% Graphite Fiber-Epoxy Matrix, $\pm 45^{\circ}$ Layup; Δ = 4 cm, 8-layer Model; t = 4 μsec, a = 2 cm)

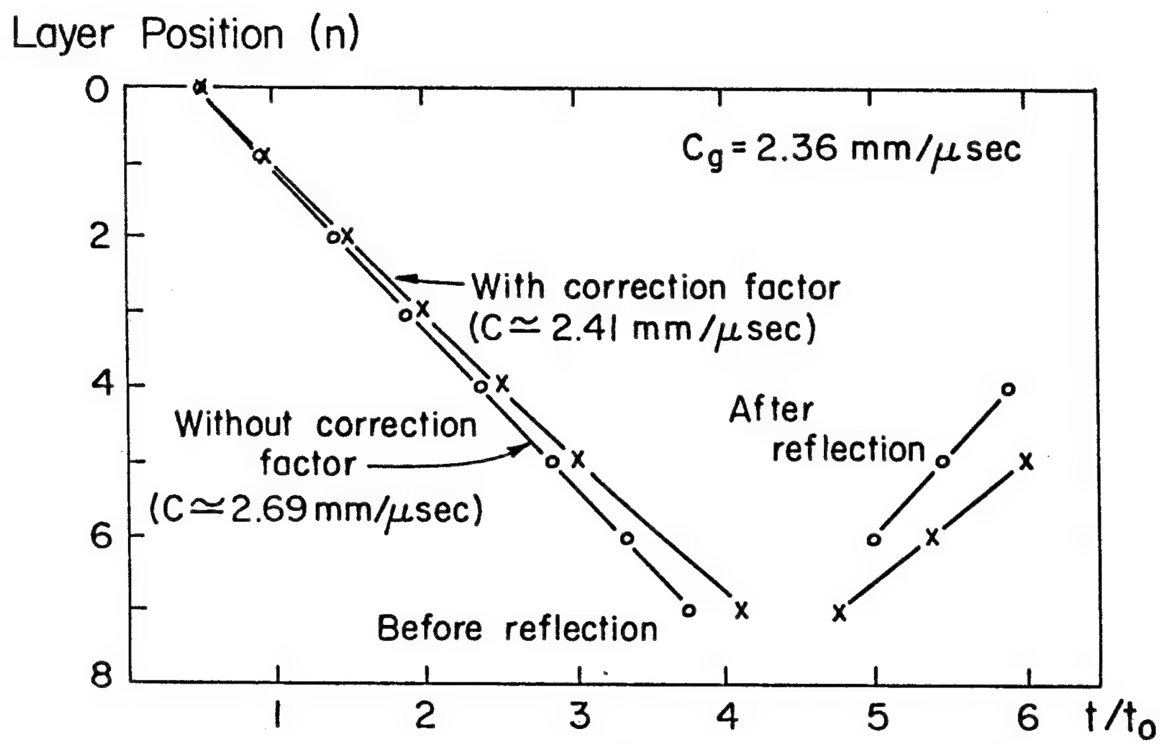


Figure 13 Effect of correction factors on transverse propagation of max $\sigma_n(x_1 = 0)$; 8-layer Model. (55% Graphite Fiber-Epoxy Matrix, $\pm 45^\circ$ Layup; Δ = 4 cm, t_o = 4 μsec, a = 2 cm)

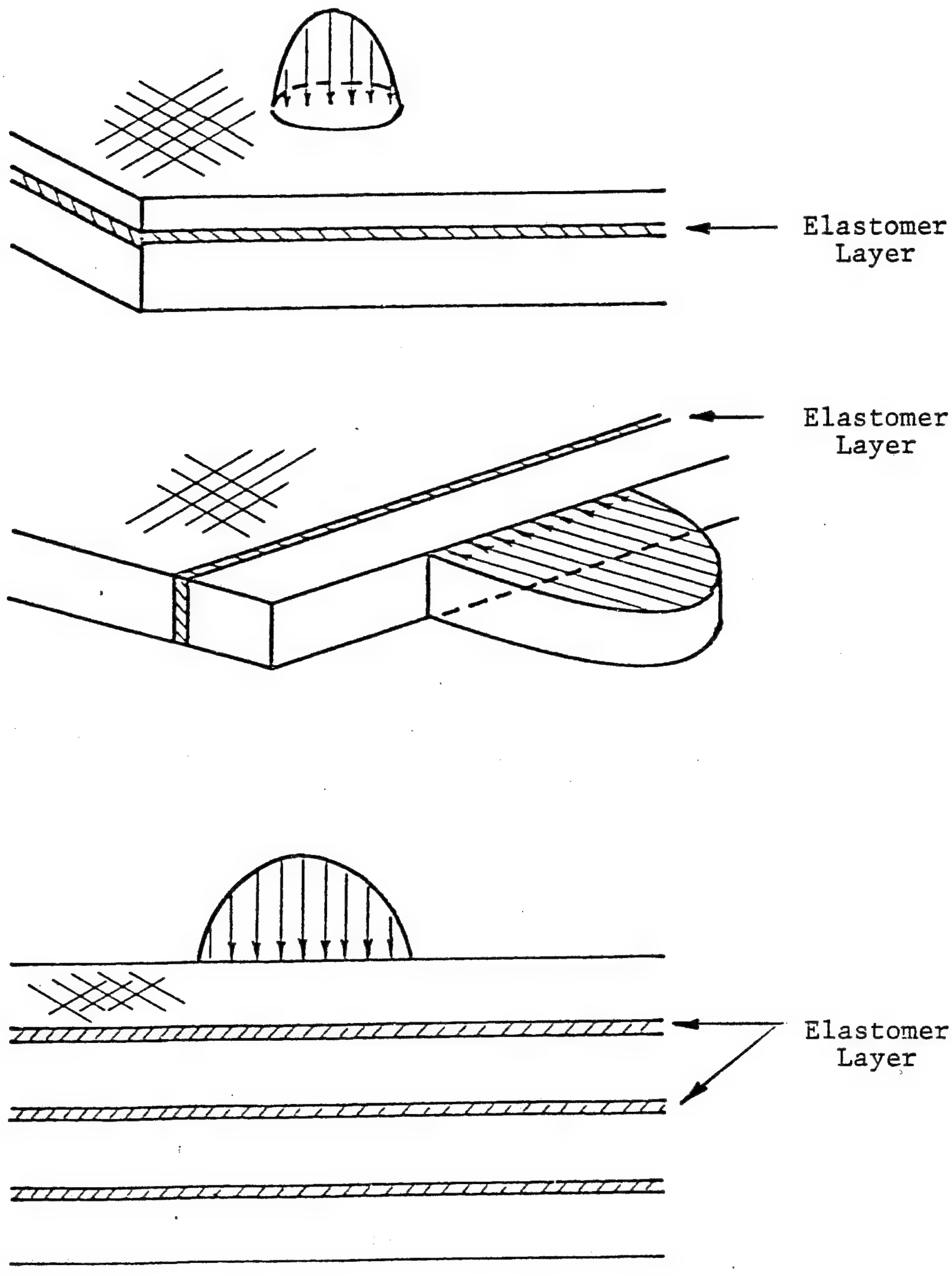


Figure 14. Viscoelastic Impact Energy Absorbing Models

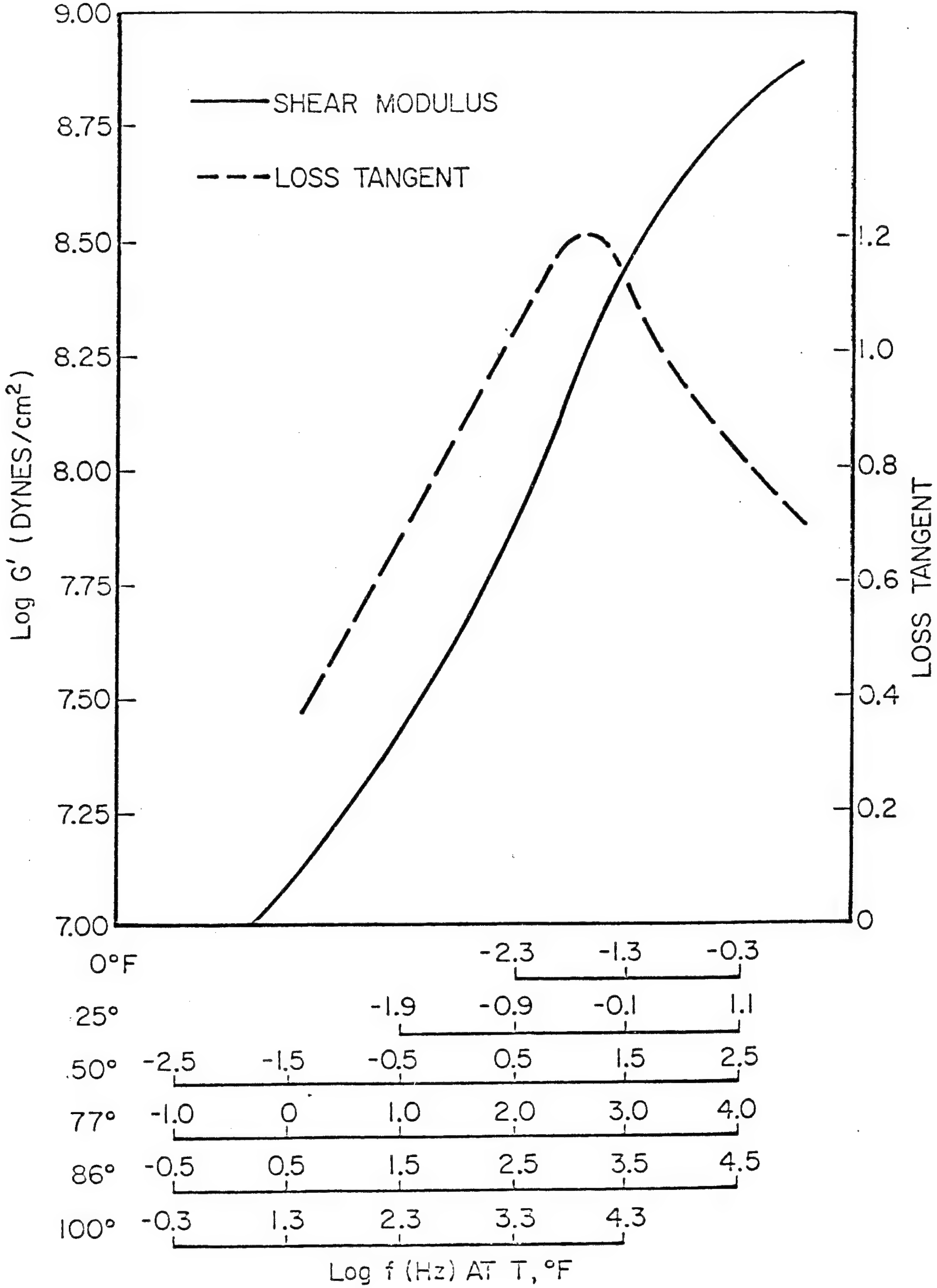


FIGURE 15a SHEAR MODULUS AND LOSS TANGENT AT T °F vs FREQUENCY FOR DUPONT LR3-604

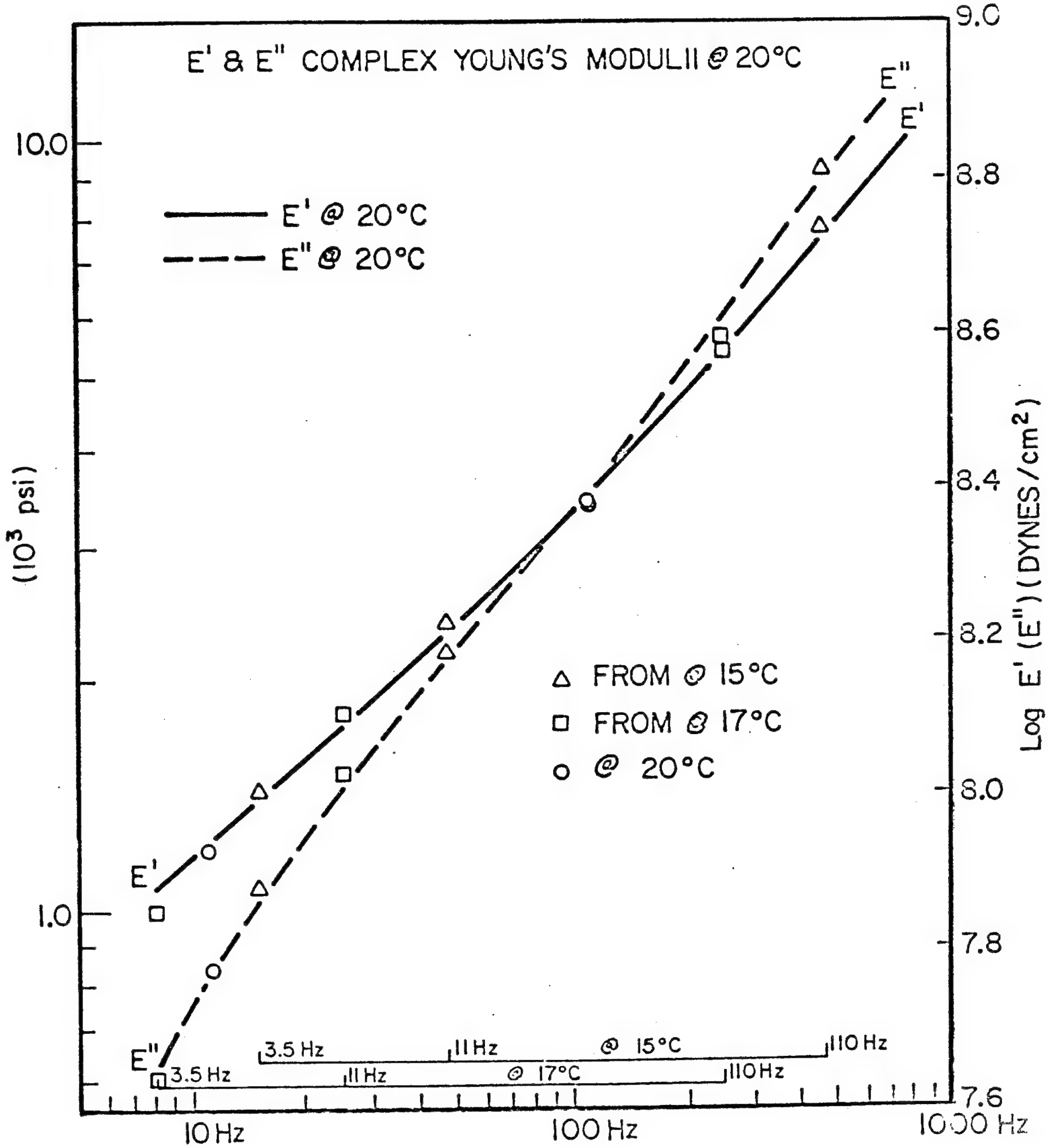


FIGURE 15b. THE COMPLEX YOUNG'S MODULUS OF LR3-604 MEASURED

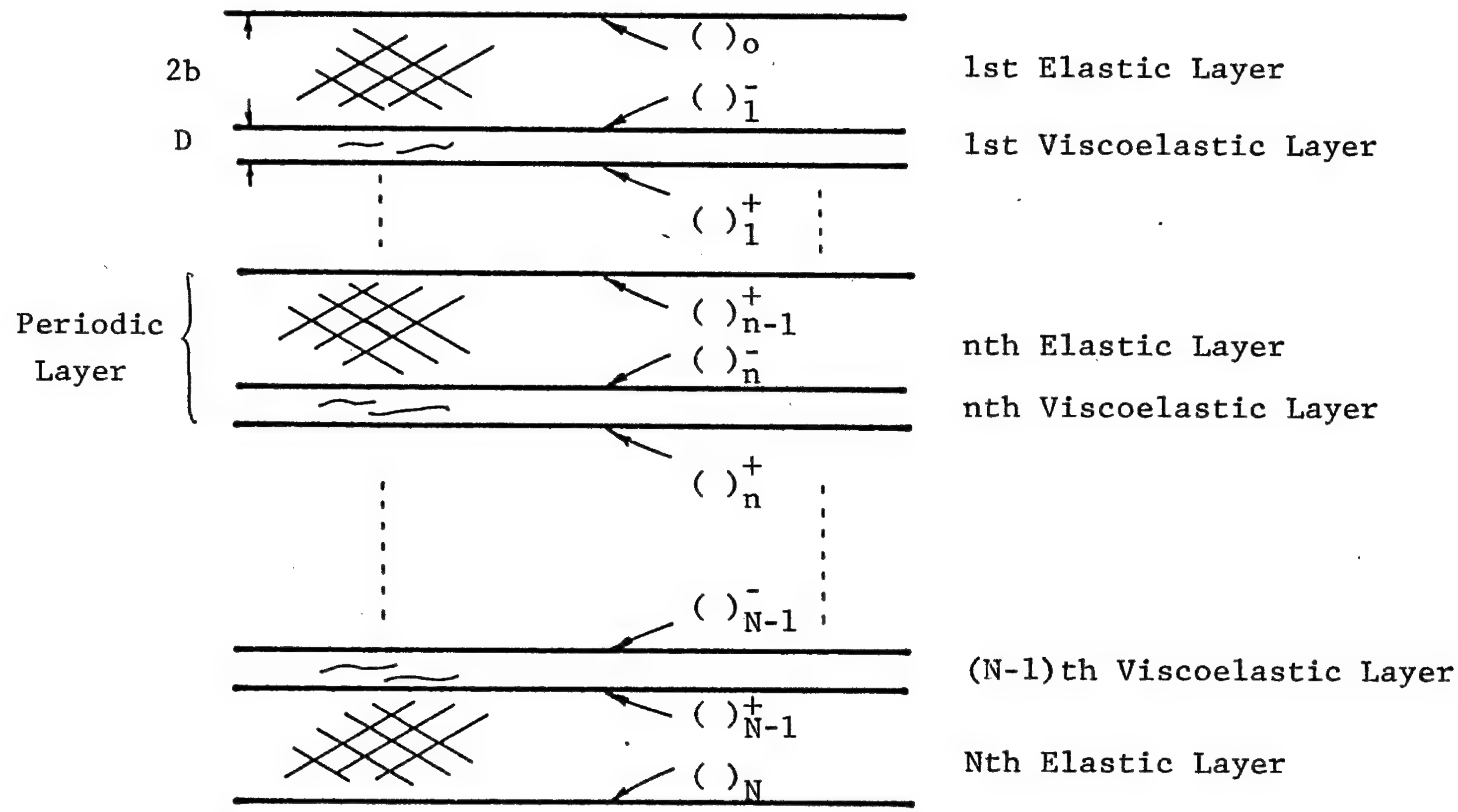


Figure 16. Plate with Viscoelastic Layers

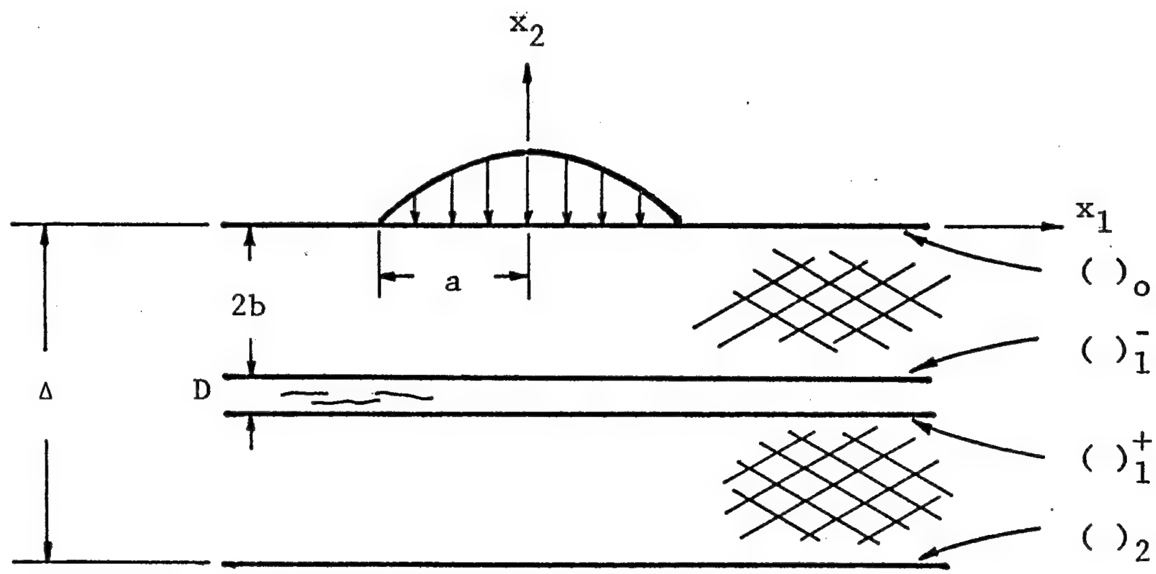


Figure 17. Impact of Plate Made of 2 Elastic Layers and a Viscoelastic Layer

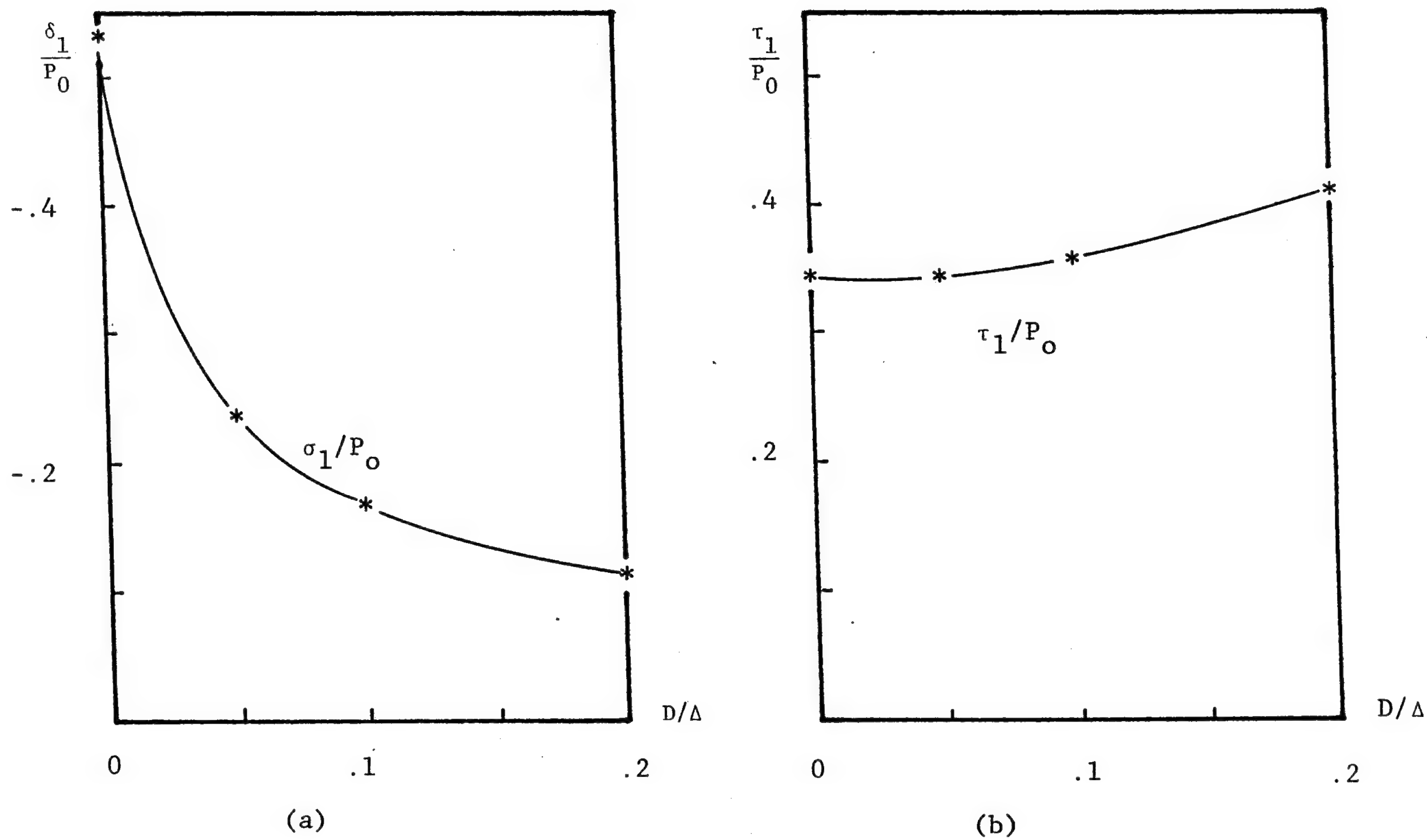


Figure 18. Peak Value of Interlaminar Stress Vs. Elastomer Thickness (two elastic layers and a viscoelastic Layer; Δ = 1 cm, t_o = 10 μsec, a = 4 cm)

*Calculated values

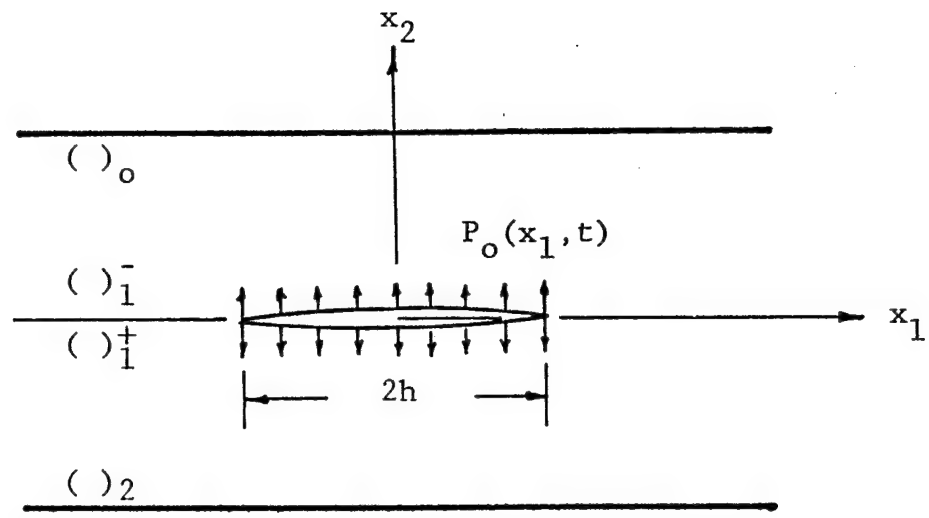


Figure 19. Composite Plate with Crack

APPENDIX A FLOW CHART

In this flow chart and program, U(I,J), V(I,J), TAU(I,J) SIGMA (I,J) and SIGMA1(I,J) represent $\hat{\hat{U}}$, $\hat{\hat{V}}$, $\hat{\hat{T}}$ and $\hat{\hat{\Sigma}}$ in Eq. (II-17,18) and integral transform of σ_{11}

(i) Define the variables (U,V,TAU,SIGMA) for displacement and stress fields in matrix form

(ii) Define other working matrices

(iii) Supply the core space for the matrices

↓

(i) Read the elastic moduli of the composite plate (in PSI) and write them

(ii) Normalize them by c_{66}

(iii) Introduce correction factors for c_{22} and c_{66}

(iv) Calculate E_{66} (c_{66} in MKS unit) and store

↓

(i) Read and write the data for geometry of composite plate.

NLAYER: Number of layers in plate

DELTA: Thickness of the plate (in cm)

RHO: Density of the composite (in gr/cm^3)

(ii) Calculate the B (half of the layer thickness)

(iii) Convert all quantities in MKS unit

↓

(i) Read and write all the impact data

NA: Impact radius by integer multiple of plate thickness (A = NA* DELTA)

TAUO: Impact time (in second)

(ii) Read and write all the data of integral steps

NX: Total step number in space

NT: Total step number in time

NXIMP: Step number in impact radius

NTIMP: Step number in impact time

(iii) Calculate step size in time and space

Space: DX = A/NXIMP

Time: DT = TAUO/NTIMP

↓

↓

(i) Calculate normalization units

Space: UNITX = DELTA (Thickness of the plate)

Time: UNITT = $A/\sqrt{E_{66}/RHO}$ (Time refuired for the quasi-shear wave to travel the impact radius)

(ii) Normalized all quantities by UNITT and UNITX

(iii) Calculate integral limits (ω_o and k_o) in Fast Fourier Transform

↓

Calculate QO(I,J), CBX(I,J), CAX(I,J), XIBX(I,J), YIAX(I,J) .... Y3AX(I,J), over a half of the range of integration (I = 1 ~ NT, J = 1, NX/2).

QO: Impact function given in Eq. (II-22)

CBX, CAX: $\cos\beta$ and $\cos\alpha$ in Eq. (II-16,17) by DPHASE

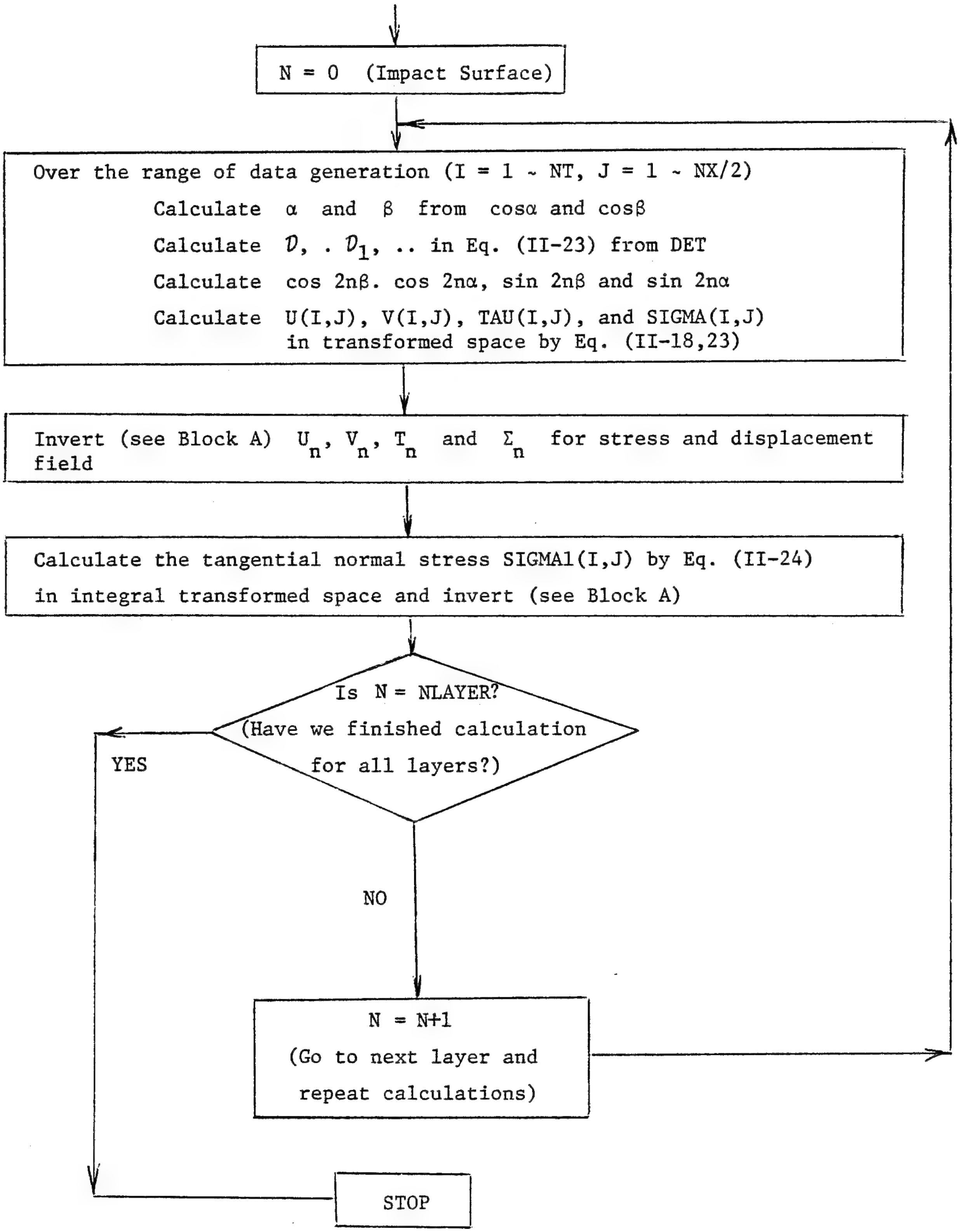
X1BX, X2BX ...: $X_1(\beta)$, $X_2(\beta)$ in Eq. (II-18,19) by DELL

Y1AX, Y2AX ...: $Y_1(\alpha)$, $Y_2(\alpha)$ in Eq. (II-18,19) by DELL

↓

Invert (see Block A) and check the impact function σ_o with QO

↓



N = 0 (Impact Surface)
Over the range of data generation (I = 1 ~ NT, J = 1 ~ NX/2)
Calculate α and β from $\cos\alpha$ and $\cos\beta$
Calculate $\mathcal{D}$, . $\mathcal{D}_1$, .. in Eq. (II-23) from DET
Calculate cos $2n\beta$. cos $2n\alpha$, sin $2n\beta$ and sin $2n\alpha$
Calculate U(I,J), V(I,J), TAU(I,J), and SIGMA(I,J) in transformed space by Eq. (II-18,23)
Invert (see Block A) U_n, V_n, T_n and Σ_n for stress and displacement field
Calculate the tangential normal stress SIGMA1(I,J) by Eq. (II-24) in integral transformed space and invert (see Block A)
Is N = NLAYER?
(Have we finished calculation for all layers?)
YES
NO
N = N+1
(Go to next layer and repeat calculations)
STOP

Block A Inversion

(i) Data xx(I,J) in integral transformed space are generated for a half of the inverse transform range: I = 1 ~ NT, J = 1 ~ NX/2

(ii) Generate full data over J = 1 ~ NX by FLIP

Symmetric flip: V(I,J), SIGMA(I,J), SIGMA1(I,J), Q0(I,J)

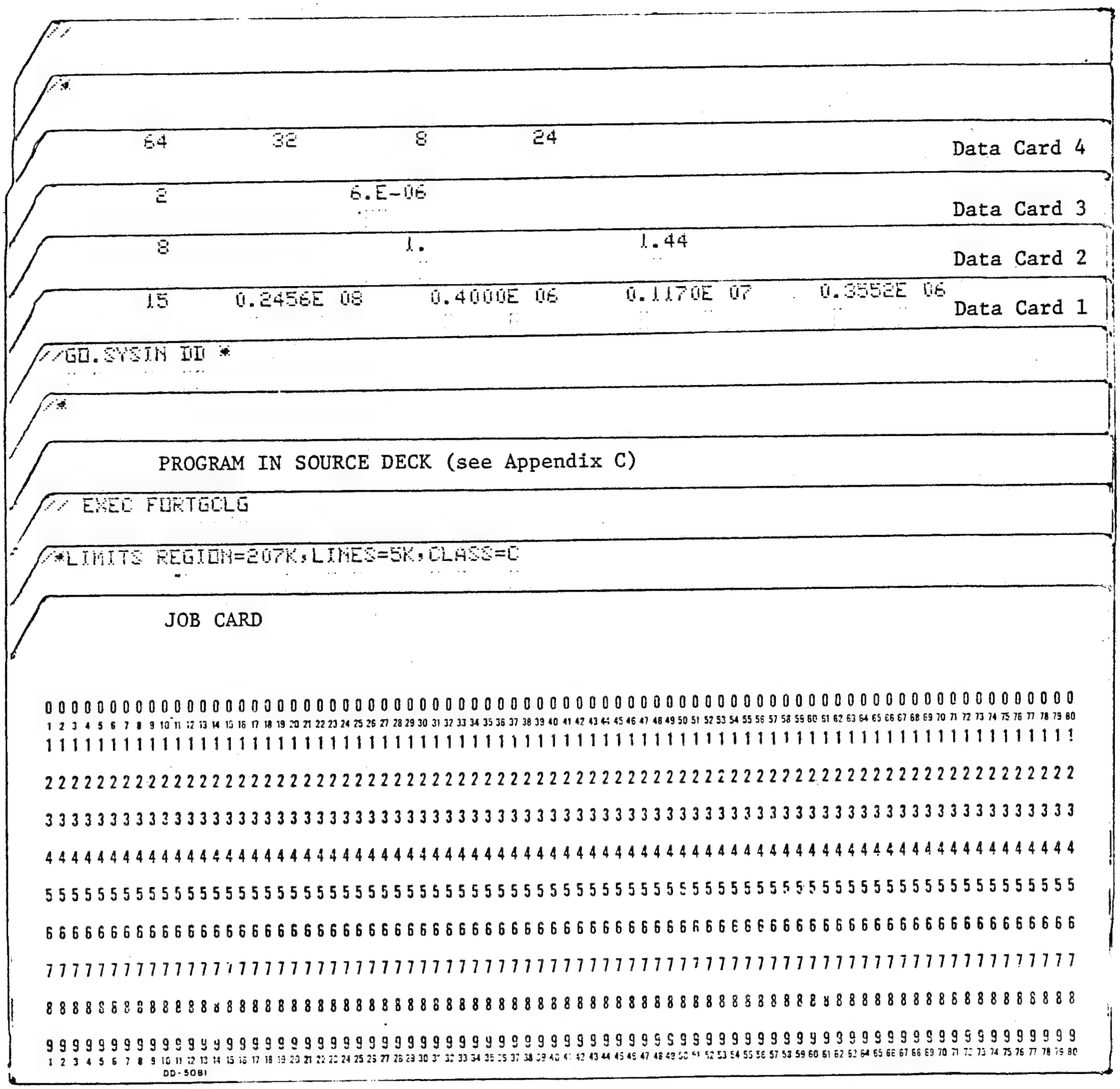
Antisymmetric flip: U(I,J), TAU(I,J)

(iii) Invert them for displacement and stress fields by FOURT

(iv) Take care of the coordinate shifts and multiplication factors in FOURT by FACT

(v) Print out by MAP

APPENDIX B SAMPLE COMPUTER DECK



```
C*******************************************************************************
C
C
C     THIS PROGRAM CALCULATES THE TRANSIENT PROPAGATION OF STRESS WAVE
C     IN A LAMINATED COMPOSITE PLATE DUE TO A NORMAL IMPACT.
C
C                ----------------------------------------
C
C     THE PRESENT PROGRAM IS A PART OF THE RESEARCH PROJECT OF
C        PROFESSOR FRANCIS C. MOON
C        DEPARTMENT OF THEORETICAL AND APPLIED MECHANICS
C        CORNELL UNIVERSITY
C     AND SUPPORTED BY NASA-LEWIS RESEARCH CENTER.
C
C
C*******************************************************************************
C
C
C     FOLLOWING USER'S GUIDES ARE PROVIDED BY DR. B.S.KIM AND ALL THE EQUATION
C     NUMERS CORRESPOND EQUATIONS IN THE ACCOMPANYING TECHNICAL REPORT
C
C_______________________________________________________________________________
C
C  1. DATA TO BE SUPPLIED BY 4 DATA CARDS (IN THE ORDER OF READING)
C
C
C        IANGLE: FIBER LAYUP ANGLE IN COMPOSITES
C        C11,C12.. : ELASTIC MODULI OF COMPOSITE LAYER(IN PSI)
C
C        NLAYER: NUMBER OF THE LAYERS IN THE GIVEN PLATE
C        D,DELTA: THICKNESS OF COMPOSITE PLATE(IN CM)
C        RHO: MASS DENSITY OF COMPOSITE LAYER(IN GR/CM**3)
C
C        NA: RADIUS OF THE IMPACT AS A MULTIPLE OF THE PLATE THICKNESS
C        TAUO: IMPACT TIME(IN SECOND)
C
C        NX: NUMBER OF INEGRATION STEPS IN SPACE
C        NT: NUMBER OF INTEGRATION STEPS IN TIME DOMAIN
C        NXIMP: NUMBER OF SPACE STEPS IN IMPACT RADIUS
C        NTIMP: NUMBER OF TIME STEPS IN IMPACT TIME
C
C
C                ----------------------------------------
C
C
C  2. WITH THE ABOVE DATA FOLLOWING PRIMARY DATA ARE CALCULATED
C
C
```

```
      C         B: A HALF OF THE LAYER THICKNESS
      C
      C         K: WAVE NUMBER FOR FOURIER TRANSFORM
      C         S: LAPLACE TRANSFORM VARIABLE
      C
      C         KO: LIMIT OF INTEGRATION FOR INVERSE FOURIER TRANSFORM FOR X
      C         OMEGAO: LIMIT OF INTEGRATION FOR INVERSE TRANSFORM IN TIME
      C         CO: LAPLACE TRANSFORM PARAMETER
      C
      C
      C                ----------------------------------------
      C
      C
      C   3. DISPLACEMENT AND STRESS FIELDS ARE CALCULATED IN TRANSFORMED SPACE
      C      AND BY INVERSIONS THESE BECOME DISPLACEMENTS AND STRESSES.
      C      THEY ARE GIVEN IN A MATIX FORM AS XX(I,J) REPRESENTING QUANTITY AT
      C      ITH TIME STEP AND JTH SPACE STEP IN X
      C
      C         U(I,J): HRIZONTAL DISPLACEMENT
      C         V(I,J): VERTICAL DISPLACEMENT
      C         SIGMA(I,J): NORMAL STRESS
      C         TAU(I,J): SHEAR STRESS
      C         SIGMA1(I,J): TANGENTIAL NORNAL STRESS
      C
      C
      C                ----------------------------------------
      C
      C
      C   4. FOLLOWINGS ARE WORKING MATRICES FOR THIS PROGRAM
      C
      C
      C         DATA(I,J), SUB(I,J): WORKING MATRICES FOR SUBROUTINE FOURT
      C         QO(I,J): INTEGRAL TRANSFORM OF IMPACT FUCTION GIVEN IN EQ(II-22)
      C
      C         CAX(I,J), CBX(I,J): COS(ALPHA) AND COS(BETA) IN EQ(II-16)
      C         X1BX(I,J), Y1AX(I,J),..: X1(BETA), Y1(ALPHA),.. IN EQ(II-18)
      C
      C
      C                ----------------------------------------
      C
      C
      C   5. FOLLOWING SUBROUTINE ARE SUPPLIED IN THE PRESENT PROGLAM
      C
      C
      C         DPHASE: CALCULATES COS(BETA) AND COS(ALPHA) IN EQ(II-16) WITH GIVEN
      C                 VALUES OF WAVE NUMBER K AND LAPLACE TRANSFORM VARIABLE S
      C
      C         DELL: CALCULATES X1(BETA), Y1(ALPHA),.. IN EQ (II-19,20)
```

```
C
C          DET: CALCULATE D,D1,.. IN EQ(II-23)
C
C          FLIP: THIS PROGLAM GENERATES ONLY A HALF OF THE DATA (X>0) DUE
C                TO SYMMETRY AND FLIP THEM TO FIND FULL DATA ALONG X AXIS
C
C          MAP: CONTROLLS THE PRINTOUT FORMAT
C
C          FACT: TAKES CARES OF THE COORDINATE SHIFT AND MULTIPLICATION FACTORS
C                IN FOURIER-LAPLACE DOUBLE INTEGRAL TRANSFORMS
C
C          FOURT: IS FAST FOURIER TRANSFORM ROUTINE SUPPLIED BY IBM
C
C
C****************************************************************************
C
C
      IMPLICIT REAL*8(K)
      COMPLEX DATA(32,64),SUB(32,32),CAX(32,32),CBX(32,32),Q0(32,32)
      COMPLEX SIGMA(32,32),SIGMA1(32,32),TAU(32,32)
      COMPLEX U(32,32),V(32,32),VX(32,32)
      COMPLEX Y1AX(32,32),Y2AX(32,32),Y3AX(32,32)
      COMPLEX X1BX(32,32),X2BX(32,32),X3BX(32,32)
      DIMENSION NN(2),MM(32)
C
      REAL*8 C11,C12,C22,C66,CHAT,E66,V66,PI,P2
      REAL*8 DCOS,DSIN,DBLE,DSQRT,DFLOAT,DLOG
      REAL*8 DELTA,D,RHO,FN,F,KX(32),B,TAU0,A,DX,DT,UNITT,UNITX
      REAL*8 Q,OMEGA0,BL,C0
C
      COMPLEX*16 CDEXP,CDLOG,CDSQRT,CDCOS,CDSIN
      COMPLEX*16 BETA,ALPHA,CB,CA,SB,SA,C2NB,C2NA,S2NB,S2NA
      COMPLEX*16 S,SI,S2,D1,D2,D3,D4
      COMPLEX*16 Y1A,Y2A,Y3A,X1B,X2B,X3B
C
      COMMON Y1A,Y2A,Y3A,X1B,X2B,X3B,D1,D2,D3,D4,C11,C12,C22,C66,CHAT
      EQUIVALENCE (DELTA,D)
      EQUIVALENCE (DATA(1,1),SUB(1,1))
C
      SI=(0.D 00,1.D 00)
      PI=3.1415926536D 00
      P2=PI*2.D 00
      INDEX=0
      INDEX=1
C
      WRITE(6,4)
4     FORMAT('1'///////////////20X,'***  WAVE PROPAGATION IN COMPOSITE P
     1LATE  ***')
```

```
      WRITE(6,5)
5     FORMAT(22X,'GRAPHITE FIBER(55%)-EPOXY MATRIX COMPOSITE')
C
C*****************************************************************************
C
C
C   INPUT DATA FOR ELASTIC PROPERTIES OF COMPOSITE PLATE
C   ALL THE DATA ARE SUPPLIED IN PSI UNIT BUT NORMALIZED BY C66
C   WHICH IS CONSTANT REGARDLESS THE LAYUP ANGLE
C
C
      READ (5,101) IANGLE,C11,C12,C22,C66
101   FORMAT(I10,4D15.7)
      CHAT=C11-C12**2/C22
      IF (IANGLE.EQ.100) GO TO 200
      WRITE(6,102) IANGLE,C11,C12,C22,C66
102   FORMAT(//   20X,'LAYUP ANGLE=',I3,3X,'DEGREE'
     $ /20X,'C(1,1)=',D12.5,'  PSI',10X,'C(1,2)=',D12.5,'  PSI'
     $ /20X,'C(2,2)=',D12.5,'  PSI',10X,'C(6,6)=',D12.5,'  PSI'//)
      GO TO 201
200   CONTINUE
      WRITE(6,210) C11,C12,C22,C66
210   FORMAT(/20X,' PLATE IS ISOTROPIC WITH POISSON''S RATIO 1/4'
     $ /20X,'C(1,1)=',D12.5,'  PSI',10X,'C(1,2)=',D12.5,'  PSI'
     $ /20X,'C(2,2)=',D12.5,'  PSI',10X,'C(6,6)=',D12.5,'  PSI'//)
201   CONTINUE
C
      E66=C66*6892.2D 00
      C11=C11/C66
      C12=C12/C66
      C22=C22/C66
      CHAT=CHAT/C66
      C66=1.D 00
C
C****** WITH CORRECTION FACTOR
C
      C66=PI**2/12.D 00
      C22=C22*C66
C
C
C     ----------------------------------
C
C   INPUT DATA FOR GEOMETRY OF COMPOSITE PLATE
C   ALL THE DATA ARE FIRST SUPPLIED IN CGS UNIT BUT CONVERTED INTO MKS UNIT
C
C
      READ(5,120) NLAYER,DELTA,RHO
120   FORMAT(I10,2D20.10)
```

```
      NL1=NLAYER+1
      FN=DFLOAT(NLAYER)
      B=DELTA/FN
      WRITE(6,121) DELTA,RHO,NLAYER,B
121   FORMAT(20X,'TOTAL THICKNESS OF COMPOSITE PLATE ; DELTA=',F10.5,
     $ '  CM'/
     $ 20X,'DENSITY OF COMPOSITE ; RHO=',F10.5,'  GR/CM**3'/
     $ 20X, 'PLATE IS MADE OF ',I3,3X,  'IDENTICAL LAYERS'/
     $ 20X,'LAYER THICKNESS ; 2B=',F10.5,'  CM'//)
C
      DELTA=DELTA/100.D 00
      RHO=RHO*1000.D 00
      B=B/200.D 00
C
C
C     ----------------------------------
C
C    INPUT DATA FOR IMPACT
C
C
      READ(5,60) NA,TAUO
60    FORMAT(I10,D20.10)
      READ(5,111) NX,NT,NXIMP,NTIMP
111   FORMAT(4I10)
C
      WRITE(6,112) NX,NXIMP,NT,NTIMP
112   FORMAT(20X,'TOTAL SPACE STEPS;  NX=',I3,5X,'WITH',I3,2X,'STEPS FOR
     $ CONTACT RADIUS'/
     $20X,'TOTAL TIME STEPS ;  NT=',I3,5X,'WITH',I3,2X,'STEPS FOR CONTAC
     $T TIME'//)
C
      A=DFLOAT(NA)*DELTA
      DX=A/DFLOAT(NXIMP)
      DT=TAUO/DFLOAT(NTIMP)
C
      WRITE(6,61) A,DX,TAUO,DT
61    FORMAT(20X,'CONTACT RADIUS ; A=',D12.5,' M'/
     $20X,'SPACE STEP ; DX=',D12.5,' M'/
     $20X,'CONTACT TIME ; TAUO=',D12.5,'  SECOND'/
     $20X,'TIME STEP ; DT=',D12.5,'  SECOND'//)
C
C
C     ----------------------------------
C
C    NORMALIZE ALL THE INPUT DATA
C
C
      V66=DSQRT(E66/RHO)
```

```
      UNITT=A/V66
      UNITX=DELTA
      F=D*D*RHO/(E66*UNITT**2)/2.D 00/FN
C
      NX2=NX/2
      NN(1)=NT
      NN(2)=NX
      DX=DX/UNITX
      DT=DT/UNITT
      KO=PI/DX
      OMEGAO=PI/DT
      BL=A/D
      CO=DLOG(1.D 06*2.D 00*DT)/(3.D 00*DT*DFLOAT(NT))
C
C
C
C*****************************************************************************************
C
C
C    CALCULATE THE IMPACT INPUT FUNCTION QO(I,J) IN EQ(II-22)
C    CALCULATE COS(BETA) AND COS(ALPHA) IN EQ(II-16) BY SUBROUTINE DPHASE
C    CALCULATE X1(BETA), Y1(ALPHA) ,.. BY SUBROUTINE DELL
C
C
      DO 30 J=1,NX2
      K=2.D 00*KO*(DFLOAT(J)-.5)/DFLOAT(NX)-KO
      K2=K**2
      KX(J)=K
      Q=PI**2/DSQRT(P2)*DSIN(K*BL)/K/((K*BL)**2-PI**2)
C
      DO 30 I=1,NT
      S=CO+SI*OMEGAO*(1.D 00-(DFLOAT(I)-.5D 00)*2.D 00/DFLOAT(NT))
      S2=S**2*F
      QO(I,J)=Q/2./S*(1.D 00-CDEXP(-S*TAUO/UNITT))*(P2*UNITT)**2
     $  /((S*TAUO)**2+(P2*UNITT)**2)
C
      CALL DPHASE (K,S2,CB,CA,NLAYER)
      CBX(I,J)=CB
      CAX(I,J)=CA
C
      CALL DELL(K,S2,CB,CA,SI,NLAYER)
      Y1AX(I,J)=Y1A
      Y2AX(I,J)=Y2A
      Y3AX(I,J)=Y3A
      X1BX(I,J)=X1B
      X2BX(I,J)=X2B
      X3BX(I,J)=X3B
C
```

```
 30    CONTINUE
 C
 C
 C
 C**************************************************************************
 C
 C
 C   REPRODUCE THE IMPACT FUNCTION TO CHECK INPUT
 C
       DO 300 I=1,NT
       DO 300 J=1,NX2
 300   SUB(I,J)=Q0(I,J)
       CALL FLIP(DATA,NX,NX2,NT,+1)
       CALL FOURT(DATA,NN,2,-1,1,0)
       CALL FACT(DATA,NX,NT,CO,OMEGAO,KO,PI,SI)
       WRITE(6,301)
 301   FORMAT('1'///////////////20X, '***  REPRODUCTION OF IMPACT FUNCTIO
      1N  ***')
       CALL MAP(DATA,NX,NT,NX2,MM,INDEX)
 C
 C
 C
 C**************************************************************************
 C
 C
 C   THIS IS THE MAIN PART OF THE PROGRAM.
 C
 C   CALCULATE D(BETA),DBAR(ALPHA),.. IN EQ(II-19,20) BY SUBROUTINE DET
 C   NEXT CALCULATE 1,V,.. IN EQ(II-18) IN TRANSFORMED SPACE
 C   AND FLIP TO FIND FULL DATA AND INVERT THEM BY MEANS OF FOURT.
 C   REPEAT THIS PROCESS FROM N=0 TO NLAYER
 C
       DO 11 N=1,NL1
       NY=N-1
       NYY=NY-1
 C
 C
 C     -----------------------------------
 C
 C   GENERATION OF DATA FOR DISPLACEMENTS AND STRESS IN TRANSFORMED SPACE
 C
       DO 100 J=1,NX2
       DO 100 I=1,NT
       CB=CBX(I,J)
       CA=CAX(I,J)
       X1B=X1BX(I,J)
       X2B=X2BX(I,J)
       X3B=X3BX(I,J)
```

```
      Y1A=Y1AX(I,J)
      Y2A=Y2AX(I,J)
      Y3A=Y3AX(I,J)
C
      SB=CDSQRT(1.D 00-CB**2)
      SA=CDSQRT(1.D 00-CA**2)
      BETA=CB+SI*SB
      ALPHA=CA+SI*SA
      BETA=CDLOG(BETA)/SI
      ALPHA=CDLOG(ALPHA)/SI
C
      CALL DET(ALPHA,BETA,SI,FN)
      C2NB=CDCOS(2.D 00*BETA*DFLOAT(NY))
      S2NB=CDSIN(2.D 00*BETA*DFLOAT(NY))
      C2NA=CDCOS(2.D 00*ALPHA*DFLOAT(NY))
      S2NA=CDSIN(2.D 00*ALPHA*DFLOAT(NY))
C
      U(I,J)=(X1B*(D1*C2NB+SI*D2*S2NB)+Y1A*(D4*C2NA+SI*D3*S2NA))*Q0(I,J)
      V(I,J)=(X2B*(D2*C2NB+SI*D1*S2NB)+Y2A*(D3*C2NA+SI*D4*S2NA))*Q0(I,J)
      TAU(I,J)=(X3B*(D2*C2NB+SI*D1*S2NB)+(D3*C2NA+SI*D4*S2NA))*Q0(I,J)
      SIGMA(I,J)=((D1*C2NB+SI*D2*S2NB)+Y3A*(D4*C2NA+SI*D3*S2NA))*Q0(I,J)
100   CONTINUE
C
C
C     ----------------------------------
C
C    INVERSION AND PRINTOUT OF HORIZONTAL DISPLACEMENT UN(I,J)
C
      DO 10 I=1,NT
      DO 10 J=1,NX2
10    SUB(I,J)=U(I,J)
      CALL FLIP(DATA,NX,NX2,NT,-1)
      CALL FOURT(DATA,NN,2,-1,1,0)
      CALL FACT(DATA,NX,NT,CO,OMEGAO,KO,PI,SI)
      WRITE(6,981) NY
981   FORMAT('1'///////////////20X,'U',I3)
      CALL MAP(DATA,NX,NT,NX2,MM,INDEX)
C
C
C     ----------------------------------
C
C    INVERSION AND PRINTOUT OF VERTICAL DISPLACEMENT VN(I,J)
C
      DO 20 I=1,NT
      DO 20 J=1,NX2
20    SUB(I,J)=V(I,J)
      CALL FLIP(DATA,NX,NX2,NT,+1)
      CALL FOURT(DATA,NN,2,-1,1,0)
```

```
      CALL FACT(DATA,NX,NT,CO,OMEGAO,KO,PI,SI)
      WRITE(6,982) NY
 982  FORMAT('1'///////////////20X,'V',I3)
      CALL MAP(DATA,NX,NT,NX2,MM,INDEX)
C
C
C     ----------------------------------
C
C    INVERSION AND PRINTOUT OF SHEAR STRESS TAU(I,J)
C
      DO 35 I=1,NT
      DO 35 J=1,NX2
 35   SUB(I,J)=TAU(I,J)
      CALL FLIP(DATA,NX,NX2,NT,-1)
      CALL FOURT(DATA,NN,2,-1,1,0)
      CALL FACT(DATA,NX,NT,CO,OMEGAO,KO,PI,SI)
      WRITE(6,983) NY
 983  FORMAT('1'///////////////20X,'TAU',I3)
      CALL MAP(DATA,NX,NT,NX2,MM,INDEX)
C
C
C     ----------------------------------
C
C    INVERSION AND PRINTOUT OF NORMAL STRESS SIGMA(I,J)
C
      DO 40 I=1,NT
      DO 40 J=1,NX2
 40   SUB(I,J)=SIGMA(I,J)
      CALL FLIP(DATA,NX,NX2,NT,+1)
      CALL FOURT(DATA,NN,2,-1,1,0)
      CALL FACT(DATA,NX,NT,CO,OMEGAO,KO,PI,SI)
      WRITE(6,984) NY
 984  FORMAT('1'///////////////20X,'SIGMA',I3)
      CALL MAP(DATA,NX,NT,NX2,MM,INDEX)
C
C
C     ----------------------------------
C
C    INVERSION AND PRINTOUT OF TANGENTIAL NORMAL STRESS SIGMA1(I,J)
C
      IF (NY.EQ.0) GO TO 160
      DO 50 I=1,NT
      DO 50 J=1,NX2
 50   SUB(I,J)=SIGMA1(I,J)+FN*C12*V(I,J)
      CALL FLIP(DATA,NX,NX2,NT,+1)
      CALL FOURT(DATA,NN,2,-1,1,0)
      CALL FACT(DATA,NX,NT,CO,OMEGAO,KO,PI,SI)
      WRITE(6,985) NYY
```

```
  985     FORMAT('1'///////////////20X,'SIGMA1',I3)
          CALL MAP(DATA,NX,NT,NX2,MM,INDEX)
C
          IF (NY.EQ.NLAYER) GO TO 70
          DO 51 I=1,NT
          DO 51 J=1,NX2
  51      SIGMA1(I,J)=-SI*KX(J)*C11*U(I,J)-FN*C12*VX(I,J)
          GO TO 80
C
  160     CONTINUE
          DO 161 I=1,NT
          DO 161 J=1,NT
  161     SIGMA1(I,J)=-SI*KX(J)*C11*U(I,J)-FN*C12*V(I,J)
          GO TO 80
C
  70      CONTINUE
          DO 71 I=1,NT
          DO 71 J=1,NX2
  71      SUB(I,J)=-SI*KX(J)*C11*U(I,J)+FN*C12*(V(I,J)-VX(I,J))
          CALL FLIP(DATA,NX,NX2,NT,+1)
          CALL FOURT(DATA,NN,2,-1,1,0)
          CALL FACT(DATA,NX,NT,CO,OMEGAO,KO,PI,SI)
          WRITE(6,985) NY
          CALL MAP(DATA,NX,NT,NX2,MM,INDEX)
          GO TO 90
C
  80      CONTINUE
          DO 81 I=1,NT
          DO 81 J=1,NX2
  81      VX(I,J)=V(I,J)
C
  90      CONTINUE
C
C
  11      CONTINUE
C
C*************************************************************************
C
C
C
C
          STOP
          END
```

```
C
C
C***********************************************************************
C
C     THIS SUBROUTINE CALCULATES THE PHASE SHIFT BETA AND ALPHA FROM
C         EQ(II-15,16) OF THE PRESENT REPORT WITH GIVEN VALUES OF
C         WAVE NUMBER K AND LAPLACE TRANSFORM VARIABLE S
C
C         CA=COS(ALPHA)
C         CB=COS(BETA)
C
C***********************************************************************
C
C
      SUBROUTINE DPHASE(K,S ,CB,CA,NLAYER)
      IMPLICIT COMPLEX*16(A,X,Y)
      COMPLEX*16 ROOTP,ROOTM,S,CDSQRT,DCMPLX,DCB,DCA
      COMPLEX*16 D1,D2,D3,D4
      COMPLEX*16 CB,CA
      REAL*8 C11,C12,C22,C66,CHAT,N,K2,DFLOAT,DBLE,K
C
      COMMON Y1A,Y2A,Y3A,X1B,X2B,X3B,D1,D2,D3,D4,C11,C12,C22,C66,CHAT
      ROOTP(AA,AB,AC)=(-AB+CDSQRT(AB**2-4.D 00*AA*AC))/(2.D 00*AA)
      ROOTM(AA,AB,AC)=(-AB-CDSQRT(AB**2-4.D 00*AA*AC))/(2.D 00*AA)
C
      N=DFLOAT(NLAYER)*2.D 00
      K2=K**2
C
C
      A1=(K2*C11/N+S)*(K2*C66/N+S)
      A2=K2*CHAT/(3.D 00*N)+N*C66+S/3.D 00-C12*C66*K2/(3.D 00*N*C22)
      A2=A2*(-C12*K2/(3.D 00*N)+N*C22+S/3.D 00)
      A3=N*C22+S/3.D 00-K2*C12*(C66*K2/N+S)/(9.D 00*N**2*C22)+C66*K2/
     $    (3.D 00*N)
      A3=A3*(C11*K2/N+S)-K2*(C12+C66)**2
      A3=A3+(C66*K2/N+S)*(CHAT*K2/(3.D 00*N)+N*C66+S/3.+C12**2*K2/
     $  (3.D 00*C22))
C
      AA=A1+A2-A3
      AB=A3-2.D 00*A2
      AC=A2
      DCB=ROOTP(AA,AB,AC)
      DCA=ROOTM(AA,AB,AC)
C
      CB=CDSQRT(DCB)
      CA=CDSQRT(DCA)
      RETURN
      END
```

```
      C
      C
      C*********************************************************************
      C
      C   THIS SUBROUTINE CALCULATES DELTA(BETA),DELTABAR(ALPHA),DELTA1(BETA),..
      C        IN EQ(II-19,20) AND X1(BETA),Y1(ALPHA) IN EQ(II-18,19,20)
      C
      C        X1B=X1(BETA),   Y1A=Y1(ALPHA)
      C              .                .
      C              .                .
      C
      C*********************************************************************
      C
      C
            SUBROUTINE DELL (K,S,CB,CA,SI,NLAYER)
            IMPLICIT REAL*8(K),COMPLEX*16(S,X,D,Y)
            COMPLEX*16 SB,CB,CDSQRT,SA,CA
            REAL*8 N,DFLOAT
            REAL*8 C11,C12,C22,C66,CHAT
      C
            COMMON Y1A,Y2A,Y3A,X1B,X2B,X3B,D1,D2,D3,D4,C11,C12,C22,C66,CHAT
      C
            K2=K**2
            N=DFLOAT(NLAYER)
      C
      C
            S11=S+C11*K2/2.D 00/N
            S66=S+C66*K2/2.D 00/N
            SA=CDSQRT(1.D 00-CA**2)
            SB=CDSQRT(1.D 00-CB**2)
      C
      C
            DELTAB=CB**3*S11*S66+SB**2*CB*(-C66*(C66+C12)*K2
           $   +S66*(S/3.D 00+CHAT*K2/6.D 00/N+2.D 00*N*C66))
            DEL1B=SI*K*SB**2*CB*(-C66-C12+C12*S66/6.D 00/N/C22)
            DEL2B=SI*SB**3*(C12*C66*K2/6.D 00/N/C22-S/3.D 00
           $  -CHAT*K2/6.D 00/N-2.D 00*N*C66)-SI*CB**2*SB*S11
            DEL3B=CB**2*SB*(C12*K*S11*S66/6.D 00/N/C22-C66*K*S11)
           $  +SB**3*(C12*K*(S/3.D 00+CHAT*K2/6.D 00/N+2.D 00*N*C66)
           $  -C12**2*C66*K*K2/6.D 00/N/C22)
      C
            DELTAA=SI*CA**2*SA*((C12+C66)*C66*K2-S66*(S/3.D 00+CHAT*K2
           $   /6.D 00/N+2.D 00*N*C66+C12**2*K2/6.D 00/N/C22))
           $   +SI*SA**3*(S/3.D 00+2.D 00*N*C22)*(C66*C12*K2/6.D 00/N/C22
           $  -S/3.D 00-CHAT*K2/6.D 00/N-2.D 00*N*C66)
            DEL1A=SA**2*CA*(-K2*C12*S66/6.D 00/6.D 00/N**2/C22+C66*K2/6.D 00/N
           $   +S/3.D 00+2.D 00*N*C22)+CA**3*S66
            DEL2A=CA**2*SA*K*(C12+C66)
```

```
      $    +SA**3*(K/6.D 00/N*(S/3.D 00+CHAT*K2/6.D 00/N+2.D 00*N*C66)
      $    -C66*C12*K**3/36.D 00/C22/N**2)
       DEL3A=-SI*CA**3*C12*K*S66+SI*SA**2*CA*(C66*K*(S/3.D 00+2.D 00*N
      $    *C22+C66*K2/6.D 00/N)-K/6.D 00/N*S66*(S/3.D 00+CHAT*K2/6.D 00
      $    /N+2.D 00*N*C66))
C
C
       X1B=-DEL1B/DELTAB
       X2B=-DEL2B/DELTAB
       X3B=-DEL3B/DELTAB
       Y1A=-DEL1A/DELTAA
       Y2A=-DEL2A/DELTAA
       Y3A=-DEL3A/DELTAA
C
       RETURN
       END
```

```
C
C
C*********************************************************************
C
C   THIS SUBROUTINE CALCUALTES D,D1,.. IN EQ(II-23) OF THE PRESENT REPORT
C
C*********************************************************************
C
C
      SUBROUTINE DET(ALPHA,BETA,SI,FN)
      IMPLICIT COMPLEX*16(D,X,Y)
      COMPLEX*16 ALPHA,BETA
      COMPLEX*16 C2NB,C2NA,S2NA,S2NB,CDSQRT,SI
      REAL*8 FN
      REAL*8 C11,C12,C22,C66,CHAT
C
      COMMON Y1A,Y2A,Y3A,X1B,X2B,X3B,D1,D2,D3,D4,C11,C12,C22,C66,CHAT
C
      C2NA=CDCOS(2.D 00*ALPHA*FN)
      S2NA=CDSIN(2.D 00*ALPHA*FN)
      C2NB=CDCOS(2.D 00*BETA*FN)
      S2NB=CDSIN(2.D 00*BETA*FN)
C
      X=Y3A*X3B*(1.D 00-C2NA*C2NB)
      Y=X3B*Y3A*S2NB*C2NA-S2NA*C2NB
C
      D=-2.D 00*X+(1.D 00+X3B**2*Y3A**2)*S2NA*S2NB
      D1=-(X-S2NA*S2NB)
      D2=-SI*Y
      D3=SI*Y*X3B
      D4=X3B*(X3B*Y3A*S2NB*S2NA+C2NA*C2NB-1.D 00)
C
      D1=D1/D
      D2=D2/D
      D3=D3/D
      D4=D4/D
C
      RETURN
      END
```

```
      C
      C
      C*********************************************************************
      C
      C
      C   ALL THE DATA IN THE MAIN PROGRAM ARE GENERATED FOR ONLY HALF OF THE
      C        PLATE WHEN X>0.  DUE TO SYMMETRY OF THE PROBLEM WE CAN GENERATE
      C        THE FULL DATA BY FLIPPING THE HALF OF THE DATA.
      C
      C
      C*********************************************************************
      C
      C
            SUBROUTINE FLIP(DATA,NX,NX2,NT,INDEX)
            COMPLEX DATA(NT,NX)
            DO 10 J=1,NX2
            JJ=NX+1-J
            DO 10 I=1,NT
            DATA(I,JJ)=FLOAT(INDEX)*DATA(I,J)
      10    CONTINUE
            RETURN
            END
```

```
      C
      C
      C*************************************************************************
      C
      C
      C    THIS SUBROUTINE TAKES CARE OF THE COORIDINATE SHIFT IN
      C         LAPLACE AND FOURIER TRANSFORM IN THE PROCESS OF APPLYING
      C         FAST FOURIER TRANSFORM ALGORITHM
      C
      C
      C*************************************************************************
      C
      C
            SUBROUTINE FACT(DATA,NX,NT,CO,WO,KO,PI,SI)
            COMPLEX DATA(NT,NX)
            COMPLEX*16 CDEXP,SI
            REAL*8 DEXP,DSQRT,DFLOAT
            REAL*8 CO,WO,PI,KO,FT,FX
            FX=DFLOAT(NX)
            FT=DFLOAT(NT)
            NX2=NX/2
      C
            DO 10 L=1,NX2
            DO 10 M=1,NT
            DATA(M,L)=DATA(M,L)*4.D 00*KO*WO/(DSQRT(2.D 00*PI)**3*FT*FX)
           $ *DEXP(CO*PI*DFLOAT(M-1)/WO)*CDEXP(SI*PI*(1.D 00-1.D 00/FX)
           $ *DFLOAT(L-1))*CDEXP(SI*PI*(1.D 00-1.D 00/FT)*DFLOAT(M-1))
      10    CONTINUE
            RETURN
            END
```

```
      C
      C*************************************************************************
      C
      C
      C   THIS SUBROUTINE CONTROLS THE FORMAT OF THE PRINTOUT OF THE FINAL RESULTS
      C
      C        IF INDEX=0: ALL THE NUMERICAL VALUES ARE PRINTED
      C                =1: THE MAXIMUM AND NORMALIZED VALUES ARE PRINTED
      C
      C
      C*************************************************************************
      C
      C
            SUBROUTINE MAP(DATA,NX,NT,NX2,MM,INDEX)
            COMPLEX DATA(NT,NX),S
            DIMENSION MM(NX2)
            IF (INDEX.EQ.1) GO TO 200
            DO 44 IQ=1,NT
      44    WRITE(6,15) IQ,(DATA(IQ,I),I=1,NX2)
      15    FORMAT(I5,2E14.5,2X,2E14.5,2X,2E14.5,2X,2E14.5/
           $     3(5X,2E14.5,2X,2E14.5,2X,2E14.5,2X,2E14.5/)/
           $     4(5X,2E14.5,2X,2E14.5,2X,2E14.5,2X,2E14.5/)/)
      200   CONTINUE
      C***  FIND THE MAXIMUM VALUE
            RS= 1.E-3
            NT5=NT-5
            NX5=NX2-5
            DO 114 I=1,NT5
            DO 114 J= 1,NX5
            S= DATA(I,J)
            TP= REAL(S)/RS
            IF (ABS(TP).LT.1.) GO TO 114
            RS= REAL(S)
      114   CONTINUE
            WRITE (6,516) RS
      516   FORMAT(20X,'***   MAXIMUM VALUE =',E12.5,'   ***'//)
            DO 119 I=1,NT5
            DO 113 J=1,NX5
            S= DATA(I,J)
      113   MM(J)= REAL(S)/RS*100
            WRITE(6,515) (MM(KIM),KIM=1,27)
      119   CONTINUE
      515   FORMAT(10X,27I3)
            RETURN
            END
```

```
      SUBROUTINE FOURT(DATA,NN,NDIM,ISIGN,IFORM,WORK)
      DIMENSION DATA(1),NN(1),IFACT(32),WORK(1)
      TWOPI=6.283185307
      IF(NDIM-1)920,1,1
    1 NTOT=2
      DO 2 IDIM=1,NDIM
      IF(NN(IDIM))920,920,2
    2 NTOT=NTOT*NN(IDIM)
C
C     MAIN LOOP FOR EACH DIMENSION
C
      NP1=2
      DO 910 IDIM=1,NDIM
      N=NN(IDIM)
      NP2=NP1*N
      IF(N-1)920,900,5
C
C     FACTOR N
C
    5 M=N
      NTWO=NP1
      IF=1
      IDIV=2
   10 IQUOT=M/IDIV
      IREM=M-IDIV*IQUOT
      IF(IQUOT-IDIV)50,11,11
   11 IF(IREM)20,12,20
   12 NTWO=NTWO+NTWO
      M=IQUOT
      GO TO 10
   20 IDIV=3
   30 IQUOT=M/IDIV
      IREM=M-IDIV*IQUOT
      IF(IQUOT-IDIV)60,31,31
   31 IF(IREM)40,32,40
   32 IFACT(IF)=IDIV
      IF=IF+1
      M=IQUOT
      GO TO 30
   40 IDIV=IDIV+2
      GO TO 30
   50 IF(IREM)60,51,60
   51 NTWO=NTWO+NTWO
      GO TO 70
   60 IFACT(IF)=M
C
C     SEPARATE FOUR CASES--
C        1. COMPLEX TRANSFORM OR REAL TRANSFORM FOR THE 4TH, 5TH,ETC.
```

```
C             DIMENSIONS.
C          2. REAL TRANSFORM FOR THE 2ND OR 3RD DIMENSION.  METHOD--
C             TRANSFORM HALF THE DATA, SUPPLYING THE OTHER HALF BY CON-
C             JUGATE SYMMETRY.
C          3. REAL TRANSFORM FOR THE 1ST DIMENSION, N ODD.  METHOD--
C             TRANSFORM HALF THE DATA AT EACH STAGE, SUPPLYING THE OTHER
C             HALF BY CONJUGATE SYMMETRY.
C          4. REAL TRANSFORM FOR THE 1ST DIMENSION, N EVEN.  METHOD--
C             TRANSFORM A COMPLEX ARRAY OF LENGTH N/2 WHOSE REAL PARTS
C             ARE THE EVEN NUMBERED REAL VALUES AND WHOSE IMAGINARY PARTS
C             ARE THE ODD NUMBERED REAL VALUES.  SEPARATE AND SUPPLY
C             THE SECOND HALF BY CONJUGATE SYMMETRY.
C
70    NON2=NP1*(NP2/NTWO)
      ICASE=1
      IF(IDIM-4)71,90,90
71    IF(IFORM)72,72,90
72    ICASE=2
      IF(IDIM-1)73,73,90
73    ICASE=3
      IF(NTWO-NP1)90,90,74
74    ICASE=4
      NTWO=NTWO/2
      N=N/2
      NP2=NP2/2
      NTOT=NTOT/2
      I=3
      DO 80 J=2,NTOT
      DATA(J)=DATA(I)
80    I=I+2
90    I1RNG=NP1
      IF(ICASE-2)100,95,100
95    I1RNG=NP0*(1+NPREV/2)
C
C     SHUFFLE ON THE FACTORS OF TWO IN N.  AS THE SHUFFLING
C     CAN BE DONE BY SIMPLE INTERCHANGE, NO WORKING ARRAY IS NEEDED
C
100   IF(NTWO-NP1)600,600,110
110   NP2HF=NP2/2
      J=1
      DO 150 I2=1,NP2,NON2
      IF(J-I2)120,130,130
120   I1MAX=I2+NON2-2
      DO 125 I1=I2,I1MAX,2
      DO 125 I3=I1,NTOT,NP2
      J3=J+I3-I2
      TEMPR=DATA(I3)
      TEMPI=DATA(I3+1)
```

```
      DATA(I3)=DATA(J3)
      DATA(I3+1)=DATA(J3+1)
      DATA(J3)=TEMPR
  125 DATA(J3+1)=TEMPI
  130 M=NP2HF
  140 IF(J-M)150,150,145
  145 J=J-M
      M=M/2
      IF(M-NON2)150,140,140
  150 J=J+M
C
C     MAIN LOOP FOR FACTORS OF TWO.  PERFORM FOURIER TRANSFORMS OF
C     LENGTH FOUR, WITH ONE OF LENGTH TWO IF NEEDED.  THE TWIDDLE FACTOR
C     W=EXP(ISIGN*2*PI*SQRT(-1)*M/(4*MMAX)).  CHECK FOR W=ISIGN*SQRT(-1)
C     AND REPEAT FOR W=ISIGN*SQRT(-1)*CONJUGATE(W).
C
      NON2T=NON2+NON2
      IPAR=NTWO/NP1
  310 IF(IPAR-2)350,330,320
  320 IPAR=IPAR/4
      GO TO 310
  330 DO 340 I1=1,I1RNG,2
      DO 340 J3=I1,NON2,NP1
      DO 340 K1=J3,NTOT,NON2T
      K2=K1+NON2
      TEMPR=DATA(K2)
      TEMPI=DATA(K2+1)
      DATA(K2)=DATA(K1)-TEMPR
      DATA(K2+1)=DATA(K1+1)-TEMPI
      DATA(K1)=DATA(K1)+TEMPR
  340 DATA(K1+1)=DATA(K1+1)+TEMPI
  350 MMAX=NON2
  360 IF(MMAX-NP2HF)370,600,600
  370 LMAX=MAX0(NON2T,MMAX/2)
      IF(MMAX-NON2)405,405,380
  380 THETA=-TWOPI*FLOAT(NON2)/FLOAT(4*MMAX)
      IF(ISIGN)400,390,390
  390 THETA=-THETA
  400 WR=COS(THETA)
      WI=SIN(THETA)
      WSTPR=-2.*WI*WI
      WSTPI=2.*WR*WI
  405 DO 570 L=NON2,LMAX,NON2T
      M=L
      IF(MMAX-NON2)420,420,410
  410 W2R=WR*WR-WI*WI
      W2I=2.*WR*WI
      W3R=W2R*WR-W2I*WI
```

```
      W3I=W2R*WI+W2I*WR
420   DO 530 I1=1,I1RNG,2
      DO 530 J3=I1,NON2,NP1
      KMIN=J3+IPAR*M
      IF(MMAX-NON2)430,430,440
430   KMIN=J3
440   KDIF=IPAR*MMAX
450   KSTEP=4*KDIF
      DO 520 K1=KMIN,NTOT,KSTEP
      K2=K1+KDIF
      K3=K2+KDIF
      K4=K3+KDIF
      IF(MMAX-NON2)460,460,480
460   U1R=DATA(K1)+DATA(K2)
      U1I=DATA(K1+1)+DATA(K2+1)
      U2R=DATA(K3)+DATA(K4)
      U2I=DATA(K3+1)+DATA(K4+1)
      U3R=DATA(K1)-DATA(K2)
      U3I=DATA(K1+1)-DATA(K2+1)
      IF(ISIGN)470,475,475
470   U4R=DATA(K3+1)-DATA(K4+1)
      U4I=DATA(K4)-DATA(K3)
      GO TO 510
475   U4R=DATA(K4+1)-DATA(K3+1)
      U4I=DATA(K3)-DATA(K4)
      GO TO 510
480   T2R=W2R*DATA(K2)-W2I*DATA(K2+1)
      T2I=W2R*DATA(K2+1)+W2I*DATA(K2)
      T3R=WR*DATA(K3)-WI*DATA(K3+1)
      T3I=WR*DATA(K3+1)+WI*DATA(K3)
      T4R=W3R*DATA(K4)-W3I*DATA(K4+1)
      T4I=W3R*DATA(K4+1)+W3I*DATA(K4)
      U1R=DATA(K1)+T2R
      U1I=DATA(K1+1)+T2I
      U2R=T3R+T4R
      U2I=T3I+T4I
      U3R=DATA(K1)-T2R
      U3I=DATA(K1+1)-T2I
      IF(ISIGN)490,500,500
490   U4R=T3I-T4I
      U4I=T4R-T3R
      GO TO 510
500   U4R=T4I-T3I
      U4I=T3R-T4R
510   DATA(K1)=U1R+U2R
      DATA(K1+1)=U1I+U2I
      DATA(K2)=U3R+U4R
      DATA(K2+1)=U3I+U4I
```

```
          DATA(K3)=U1R-U2R
          DATA(K3+1)=U1I-U2I
          DATA(K4)=U3R-U4R
  520     DATA(K4+1)=U3I-U4I
          KMIN=4*(KMIN-J3)+J3
          KDIF=KSTEP
          IF(KDIF-NP2)450,530,530
  530     CONTINUE
          M=MMAX-M
          IF(ISIGN)540,550,550
  540     TEMPR=WR
          WR=-WI
          WI=-TEMPR
          GO TO 560
  550     TEMPR=WR
          WR=WI
          WI=TEMPR
  560     IF(M-LMAX)565,565,410
  565     TEMPR=WR
          WR=WR*WSTPR-WI*WSTPI+WR
  570     WI=WI*WSTPR+TEMPR*WSTPI+WI
          IPAR=3-IPAR
          MMAX=MMAX+MMAX
          GO TO 360
C
C         MAIN LOOP FOR FACTORS NOT EQUAL TO TWO.  APPLY THE TWIDDLE FACTOR
C         W=EXP(ISIGN*2*PI*SQRT(-1)*(J2-1)*(J1-J2)/(NP2*IFP1)), THEN
C         PERFORM A FOURIER TRANSFORM OF LENGTH IFACT(IF), MAKING USE OF
C         CONJUGATE SYMMETRIES.
C
  600     IF(NTWO-NP2)605,700,700
  605     IFP1=NON2
          IF=1
          NP1HF=NP1/2
  610     IFP2=IFP1/IFACT(IF)
          J1RNG=NP2
          IF(ICASE-3)612,611,612
  611     J1RNG=(NP2+IFP1)/2
          J2STP=NP2/IFACT(IF)
          J1RG2=(J2STP+IFP2)/2
  612     J2MIN=1+IFP2
          IF(IFP1-NP2)615,640,640
  615     DO 635 J2=J2MIN,IFP1,IFP2
          THETA=-TWOPI*FLOAT(J2-1)/FLOAT(NP2)
          IF(ISIGN)625,620,620
  620     THETA=-THETA
  625     SINTH=SIN(THETA/2.)
          WSTPR=-2.*SINTH*SINTH
```

```
      WSTPI=SIN(THETA)
      WR=WSTPR+1.
      WI=WSTPI
      J1MIN=J2+IFP1
      DO 635 J1=J1MIN,J1RNG,IFP1
      I1MAX=J1+I1RNG-2
      DO 630 I1=J1,I1MAX,2
      DO 630 I3=I1,NTOT,NP2
      J3MAX=I3+IFP2-NP1
      DO 630 J3=I3,J3MAX,NP1
      TEMPR=DATA(J3)
      DATA(J3)=DATA(J3)*WR-DATA(J3+1)*WI
  630 DATA(J3+1)=TEMPR*WI+DATA(J3+1)*WR
      TEMPR=WR
      WR=WR*WSTPR-WI*WSTPI+WR
  635 WI=TEMPR*WSTPI+WI*WSTPR+WI
  640 THETA=-TWOPI/FLOAT(IFACT(IF))
      IF(ISIGN)650,645,645
  645 THETA=-THETA
  650 SINTH=SIN(THETA/2.)
      WSTPR=-2.*SINTH*SINTH
      WSTPI=SIN(THETA)
      KSTEP=2*N/IFACT(IF)
      KRANG=KSTEP*(IFACT(IF)/2)+1
      DO 698 I1=1,I1RNG,2
      DO 698 I3=I1,NTOT,NP2
      DO 690 KMIN=1,KRANG,KSTEP
      J1MAX=I3+J1RNG-IFP1
      DO 680 J1=I3,J1MAX,IFP1
      J3MAX=J1+IFP2-NP1
      DO 680 J3=J1,J3MAX,NP1
      J2MAX=J3+IFP1-IFP2
      K=KMIN+(J3-J1+(J1-I3)/IFACT(IF))/NP1HF
      IF(KMIN-1)655,655,665
  655 SUMR=0.
      SUMI=0.
      DO 660 J2=J3,J2MAX,IFP2
      SUMR=SUMR+DATA(J2)
  660 SUMI=SUMI+DATA(J2+1)
      WORK(K)=SUMR
      WORK(K+1)=SUMI
      GO TO 680
  665 KCONJ=K+2*(N-KMIN+1)
      J2=J2MAX
      SUMR=DATA(J2)
      SUMI=DATA(J2+1)
      OLDSR=0.
      OLDSI=0.
```

```
          J2=J2-IFP2                                                      FFTT364
    670   TEMPR=SUMR                                                      FFTT365
          TEMPI=SUMI                                                      FFTT366
          SUMR=TWOWR*SUMR-OLDSR+DATA(J2)                                  FFTT367
          SUMI=TWOWR*SUMI-OLDSI+DATA(J2+1)                                FFTT368
          OLDSR=TEMPR                                                     FFTT369
          OLDSI=TEMPI                                                     FFTT370
          J2=J2-IFP2                                                      FFTT371
          IF(J2-J3)675,675,670                                            FFTT372
    675   TEMPR=WR*SUMR-OLDSR+DATA(J2)                                    FFTT373
          TEMPI=WI*SUMI                                                   FFTT374
          WORK(K)=TEMPR-TEMPI                                             FFTT375
          WORK(KCONJ)=TEMPR+TEMPI                                         FFTT376
          TEMPR=WR*SUMI-OLDSI+DATA(J2+1)                                  FFTT377
          TEMPI=WI*SUMR                                                   FFTT378
          WORK(K+1)=TEMPR+TEMPI                                           FFTT379
          WORK(KCONJ+1)=TEMPR-TEMPI                                       FFTT380
    680   CONTINUE                                                        FFTT381
          IF(KMIN-1)685,685,686                                           FFTT382
    685   WR=WSTPR+1.                                                     FFTT383
          WI=WSTPI                                                        FFTT384
          GO TO 690                                                       FFTT385
    686   TEMPR=WR                                                        FFTT386
          WR=WR*WSTPR-WI*WSTPI+WR                                         FFTT387
          WI=TEMPR*WSTPI+WI*WSTPR+WI                                      FFTT388
    690   TWOWR=WR+WR                                                     FFTT389
          IF(ICASE-3)692,691,692                                          FFTT390
    691   IF(IFP1-NP2)695,692,692                                         FFTT391
    692   K=1                                                             FFTT392
          I2MAX=I3+NP2-NP1                                                FFTT393
          DO 693 I2=I3,I2MAX,NP1                                          FFTT394
          DATA(I2)=WORK(K)                                                FFTT395
          DATA(I2+1)=WORK(K+1)                                            FFTT396
    693   K=K+2                                                           FFTT397
          GO TO 698                                                       FFTT398
    C                                                                     FFTT399
    C     COMPLETE A REAL TRANSFORM IN THE 1ST DIMENSION, N ODD, BY CON-  FFTT400
    C     JUGATE SYMMETRIES AT EACH STAGE.                                FFTT401
    C                                                                     FFTT402
    695   J3MAX=I3+IFP2-NP1                                               FFTT403
          DO 697 J3=I3,J3MAX,NP1                                          FFTT404
          J2MAX=J3+NP2-J2STP                                              FFTT405
          DO 697 J2=J3,J2MAX,J2STP                                        FFTT406
          J1MAX=J2+J1RG2-IFP2                                             FFTT407
          J1CNJ=J3+J2MAX+J2STP-J2                                         FFTT408
          DO 697 J1=J2,J1MAX,IFP2                                         FFTT409
          K=1+J1-I3                                                       FFTT410
          DATA(J1)=WORK(K)                                                FFTT411
```

```
      DATA(J1+1)=WORK(K+1)
      IF(J1-J2)697,697,696
  696 DATA(J1CNJ)=WORK(K)
      DATA(J1CNJ+1)=-WORK(K+1)
  697 J1CNJ=J1CNJ-IFP2
  698 CONTINUE
      IF=IF+1
      IFP1=IFP2
      IF(IFP1-NP1)700,700,610
C
C     COMPLETE A REAL TRANSFORM IN THE 1ST DIMENSION, N EVEN, BY CON-
C     JUGATE SYMMETRIES.
C
  700 GO TO (900,800,900,701),ICASE
  701 NHALF=N
      N=N+N
      THETA=-TWOPI/FLOAT(N)
      IF(ISIGN)703,702,702
  702 THETA=-THETA
  703 SINTH=SIN(THETA/2.)
      WSTPR=-2.*SINTH*SINTH
      WSTPI=SIN(THETA)
      WR=WSTPR+1.
      WI=WSTPI
      IMIN=3
      JMIN=2*NHALF-1
      GO TO 725
  710 J=JMIN
      DO 720 I=IMIN,NTOT,NP2
      SUMR=(DATA(I)+DATA(J))/2.
      SUMI=(DATA(I+1)+DATA(J+1))/2.
      DIFR=(DATA(I)-DATA(J))/2.
      DIFI=(DATA(I+1)-DATA(J+1))/2.
      TEMPR=WR*SUMI+WI*DIFR
      TEMPI=WI*SUMI-WR*DIFR
      DATA(I)=SUMR+TEMPR
      DATA(I+1)=DIFI+TEMPI
      DATA(J)=SUMR-TEMPR
      DATA(J+1)=-DIFI+TEMPI
  720 J=J+NP2
      IMIN=IMIN+2
      JMIN=JMIN-2
      TEMPR=WR
      WR=WR*WSTPR-WI*WSTPI+WR
      WI=TEMPR*WSTPI+WI*WSTPR+WI
  725 IF(IMIN-JMIN)710,730,740
  730 IF(ISIGN)731,740,740
  731 DO 735 I=IMIN,NTOT,NP2
```

```
  735   DATA(I+1)=-DATA(I+1)
  740   NP2=NP2+NP2
        NTOT=NTOT+NTOT
        J=NTOT+1
        IMAX=NTOT/2+1
  745   IMIN=IMAX-2*NHALF
        I=IMIN
        GO TO 755
  750   DATA(J)=DATA(I)
        DATA(J+1)=-DATA(I+1)
  755   I=I+2
        J=J-2
        IF(I-IMAX)750,760,760
  760   DATA(J)=DATA(IMIN)-DATA(IMIN+1)
        DATA(J+1)=0.
        IF(I-J)770,780,780
  765   DATA(J)=DATA(I)
        DATA(J+1)=DATA(I+1)
  770   I=I-2
        J=J-2
        IF(I-IMIN)775,775,765
  775   DATA(J)=DATA(IMIN)+DATA(IMIN+1)
        DATA(J+1)=0.
        IMAX=IMIN
        GO TO 745
  780   DATA(1)=DATA(1)+DATA(2)
        DATA(2)=0.
        GO TO 900
C
C       COMPLETE A REAL TRANSFORM FOR THE 2ND OR 3RD DIMENSION BY
C       CONJUGATE SYMMETRIES.
C
  800   IF(I1RNG-NP1)805,900,900
  805   DO 860 I3=1,NTOT,NP2
        I2MAX=I3+NP2-NP1
        DO 860 I2=I3,I2MAX,NP1
        IMIN=I2+I1RNG
        IMAX=I2+NP1-2
        JMAX=2*I3+NP1-IMIN
        IF(I2-I3)820,820,810
  810   JMAX=JMAX+NP2
  820   IF(IDIM-2)850,850,830
  830   J=JMAX+NP0
        DO 840 I=IMIN,IMAX,2
        DATA(I)=DATA(J)
        DATA(I+1)=-DATA(J+1)
  840   J=J-2
  850   J=JMAX
```

```
          DO 860 I=IMIN,IMAX,NP0
          DATA(I)=DATA(J)
          DATA(I+1)=-DATA(J+1)
  860     J=J-NP0
  C
  C       END OF LOOP ON EACH DIMENSION
  C
  900     NP0=NP1
          NP1=NP2
  910     NPREV=N
  920     RETURN
          END
```

```
*** WAVE PROPAGATION IN COMPOSITE PLATE ***
  GRAPHITE FIBER(55%)-EPOXY MATRIX COMPOSITE


LAYUP ANGLE= 15   DEGREE
C(1,1)= 0.24560D+08  PSI           C(1,2)= 0.40000D+06  PSI
C(2,2)= 0.11700D+07  PSI           C(6,6)= 0.35520D+06  PSI


TOTAL THICKNESS OF COMPOSITE PLATE ; DELTA=   1.00000  CM
DENSITY OF COMPOSITE ; RHO=   1.44000  GR/CM**3
PLATE IS MADE OF   8   IDENTICAL LAYERS
LAYER THICKNESS ; 2B=   0.12500  CM


TOTAL SPACE STEPS;  NX= 64     WITH  8  STEPS FOR CONTACT RADIUS
TOTAL TIME STEPS ;  NT= 32     WITH 24  STEPS FOR CONTACT TIME


CONTACT RADIUS ; A= 0.20000D-01 M
SPACE STEP ; DX= 0.25000D-02 M
CONTACT TIME ; TAU0= 0.60000D-05  SECOND
TIME STEP ; DT= 0.25000D-06  SECOND
```

*** REPRODUCTION OF IMPACT FUNCTION ***
*** MAXIMUM VALUE =-0.10003E+01 ***

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	6	5	4	3	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
14	14	12	10	7	4	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
25	24	21	17	12	7	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
37	35	31	25	18	11	5	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
50	48	42	34	25	15	7	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
62	60	53	43	31	19	9	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
75	72	64	51	37	23	11	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
85	82	72	58	42	26	12	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
93	89	79	64	46	28	13	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
98	94	83	67	49	30	14	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
100	96	85	69	50	30	14	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
98	94	83	67	49	30	14	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
93	89	79	64	46	28	13	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
85	82	72	58	42	26	12	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
75	72	64	51	37	23	11	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
62	60	53	43	31	19	9	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
50	48	42	34	25	15	7	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
37	35	31	25	18	11	5	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
25	24	21	17	12	7	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
14	14	12	10	7	4	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	6	5	4	3	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

SIGMA 1
*** MAXIMUM VALUE =-0.10067E+01 ***

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	4	4	3	2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	11	10	8	6	3	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
22	21	19	15	11	6	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
34	32	29	23	17	10	5	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
46	45	40	32	23	14	7	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
60	57	51	41	30	18	9	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
72	69	61	50	36	22	10	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
83	80	71	57	41	25	12	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
91	88	78	63	46	28	13	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
97	93	83	67	48	30	14	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
100	96	85	69	50	31	14	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
98	95	84	68	49	30	14	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
94	90	80	65	47	29	14	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
87	83	74	60	43	27	13	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
77	74	65	53	38	23	11	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
65	62	55	45	32	20	9	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
52	50	44	36	26	16	7	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
38	37	33	26	19	12	5	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
26	25	22	18	13	8	4	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	15	13	11	8	5	2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	7	6	5	4	2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	2	2	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

SIGMA 3
*** MAXIMUM VALUE =-0.96697E+00 ***

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	4	3	3	2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	10	8	7	5	3	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
19	18	16	13	9	6	3	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
29	28	25	20	15	9	4	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
41	40	35	29	21	13	6	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
54	52	46	38	27	17	8	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
67	64	57	46	34	21	10	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
78	75	67	54	40	25	12	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
88	84	75	61	44	28	14	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
95	91	81	66	48	30	15	5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
99	95	84	69	50	31	15	5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
100	96	85	69	50	32	16	5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
97	93	83	68	49	31	16	5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
91	88	78	63	46	29	15	5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
81	78	70	57	42	26	14	5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
69	67	60	49	36	23	12	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
56	54	48	40	29	19	10	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
44	42	38	31	23	15	8	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
33	32	28	23	17	12	7	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

SIGMA 5
*** MAXIMUM VALUE =-0.98567E+00 ***

1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	2	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	2	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	2	2	2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	2	2	2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	3	2	2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	3	2	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	2	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	3	3	2	2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	10	9	7	5	3	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
20	19	17	14	10	6	3	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
31	30	26	21	16	10	5	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
43	41	37	30	22	14	7	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
55	53	47	38	28	18	9	3	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
66	64	57	46	34	21	11	4	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
77	74	66	53	39	25	13	4	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
86	83	74	60	44	28	14	5	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
94	91	81	66	48	31	16	6	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
99	96	85	69	51	32	17	6	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
100	96	85	70	51	33	17	6	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
95	91	81	67	49	32	17	7	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
85	82	73	60	44	29	15	6	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

SIGMA 7
*** MAXIMUM VALUE =-0.53190E+00 ***

1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	2	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	2	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	3	2	2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	3	2	2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	3	2	2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	2	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	2	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-5	-5	-4	-3	-2	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-8	-8	-7	-6	-4	-2	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-7	-6	-6	-5	-3	-2	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	14	12	10	7	4	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
32	31	28	23	17	11	5	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
51	49	44	36	27	17	9	3	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
69	67	60	49	36	23	12	5	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
84	81	73	59	44	28	15	6	2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
95	91	81	67	49	32	17	7	2	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
100	96	85	70	51	33	17	7	2	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
97	94	83	68	50	32	17	7	2	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
89	86	76	62	45	29	15	6	1	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0

U 4

*** MAXIMUM VALUE = 0.86667E-02 ***

0	0	0	0	0	0	0	-1	-1	-1	-1	-2	-2	-2	-2	-2	-2	-2	-1	-1	-1	-1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	-1	-1	-1	-1	-2	-2	-2	-2	-2	-2	-1	-1	-1	-1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	-1	-1	-1	-1	-2	-2	-2	-2	-2	-2	-1	-1	-1	-1	0	0	0
0	0	0	1	1	1	0	0	0	0	0	0	-1	-1	-1	-1	-2	-2	-2	-2	-1	-1	-1	-1	-1	0	0
0	0	1	1	1	1	1	1	0	0	0	0	0	-1	-1	-1	-1	-1	-2	-2	-2	-1	-1	-1	-1	-1	0
0	0	1	1	2	2	1	1	1	0	0	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
0	0	1	2	2	2	2	1	1	1	0	0	0	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
0	0	1	2	2	2	2	2	1	1	1	0	0	0	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
0	0	1	1	2	2	2	2	2	1	1	1	1	0	0	0	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1
0	0	0	1	1	1	2	2	2	2	2	1	1	1	0	0	0	0	0	0	-1	-1	-1	-1	-1	-1	-1
0	0	0	0	0	0	1	1	2	2	2	2	1	1	1	0	0	0	0	0	0	0	-1	-1	-1	-1	-1
0	0	0	0	0	0	0	1	1	2	2	2	2	1	1	1	1	0	0	0	0	0	0	-1	-1	-1	-1
0	0	0	-1	0	0	0	0	1	1	2	2	2	2	1	1	1	1	0	0	0	0	0	0	-1	-1	-1
0	0	0	0	0	0	0	0	0	1	1	1	2	2	2	1	1	1	1	0	0	0	0	0	0	-1	-1
0	1	2	3	3	2	2	1	0	0	1	1	1	1	2	2	1	1	1	1	0	0	0	0	0	-1	-1
0	3	6	8	8	8	6	4	2	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	-1
0	6	11	15	16	16	13	9	6	3	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	-1
0	9	18	24	26	26	22	17	12	7	4	2	1	1	1	1	1	1	1	1	1	0	0	0	0	0	-1
0	13	25	34	38	38	34	27	20	14	9	5	3	1	1	1	1	1	1	1	1	0	0	0	0	-1	-1
0	17	33	44	50	51	47	39	30	22	15	10	6	4	2	1	1	1	1	0	0	0	0	0	0	-1	-1
0	21	39	54	62	64	60	52	42	33	24	17	11	7	4	2	1	0	0	0	0	0	0	0	-1	-1	-1
0	24	45	62	72	75	72	65	55	45	35	26	19	13	8	5	2	1	0	0	0	0	-1	-1	-1	-1	-1
0	26	49	68	80	85	83	77	67	57	46	36	27	20	13	8	5	2	0	0	-1	-1	-1	-1	-1	0	0
0	27	52	72	85	92	92	87	78	68	58	47	37	28	20	14	8	4	1	0	-1	-1	-1	-1	-1	0	0
0	27	52	73	87	95	97	94	87	78	68	58	47	37	28	20	13	8	4	1	0	-1	-1	-1	0	0	0
0	26	51	71	86	96	100	98	93	86	77	67	57	47	37	27	19	13	8	4	1	0	0	0	0	0	0
0	25	48	68	83	93	98	99	96	91	84	75	65	55	45	36	27	19	13	8	5	2	1	0	0	0	0

V 4
*** MAXIMUM VALUE = 0.18017E+00 ***

-3	-3	-3	-2	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-3	-3	-3	-2	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-3	-3	-3	-2	-2	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-4	-3	-3	-2	-2	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-4	-4	-3	-3	-2	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-4	-4	-3	-3	-2	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-4	-4	-4	-3	-2	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-4	-4	-4	-3	-2	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-5	-5	-4	-3	-3	-2	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-5	-5	-5	-4	-3	-2	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-6	-6	-5	-4	-3	-2	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-6	-6	-5	-4	-3	-2	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-5	-5	-5	-4	-3	-2	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-3	-3	-3	-3	-2	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	3	2	2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	8	7	6	4	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
16	15	13	11	7	4	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
24	23	21	16	11	7	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
34	32	29	23	16	10	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
44	42	37	30	21	13	6	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
55	52	46	37	27	16	7	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
65	63	56	45	32	20	9	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
76	73	64	52	38	23	11	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
85	82	72	59	42	26	13	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
93	89	79	64	47	29	14	5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
100	96	85	69	50	31	16	5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

SIGMA 4
*** MAXIMUM VALUE =-0.94195E+00 ***

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	2	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	2	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	3	2	2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	2	2	2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	2	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	2	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8	7	6	5	4	2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	15	13	11	8	5	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
25	24	22	17	13	8	4	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
37	35	32	26	19	12	6	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
50	48	43	35	25	16	8	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
63	60	54	44	32	20	10	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
75	72	64	52	38	24	12	4	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
86	83	73	60	44	27	14	5	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
94	90	80	65	48	30	15	5	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
99	95	84	69	50	32	16	6	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
100	96	85	69	51	32	16	6	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
98	94	84	68	50	32	16	6	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
93	90	80	65	48	30	16	6	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
88	84	75	61	45	29	15	6	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
79	76	68	56	41	27	14	6	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
68	65	58	48	36	23	13	5	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

TAU 4

*** MAXIMUM VALUE =-0.10045E+00 ***

0	-1	-2	-3	-3	-3	-2	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	-1	-2	-3	-3	-3	-2	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	-1	-3	-3	-4	-3	-2	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	-1	-3	-4	-4	-4	-3	-2	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	-1	-3	-4	-4	-4	-3	-2	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	-1	-3	-4	-5	-4	-3	-2	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	-2	-3	-5	-5	-5	-4	-3	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	-2	-4	-5	-6	-5	-4	-3	-2	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	-2	-4	-6	-6	-6	-5	-3	-2	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	-2	-5	-6	-7	-7	-5	-4	-2	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	-2	-5	-7	-7	-7	-5	-4	-2	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	-2	-4	-5	-6	-6	-5	-4	-2	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	-1	-2	-3	-3	-3	-3	-2	-2	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	1	1	1	0	0	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	3	5	7	8	7	6	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	6	11	15	17	16	13	8	3	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	9	18	24	27	25	21	14	7	4	3	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	13	25	33	37	36	30	20	11	7	6	4	3	2	1	0	0	0	0	0	0	0	0	0	0	0	0
0	17	32	42	47	46	38	26	15	10	8	6	5	3	2	1	0	0	0	0	0	0	0	0	0	0	0
0	20	39	51	57	55	46	31	18	12	10	9	7	5	3	2	1	0	0	0	0	0	0	0	0	0	0
0	24	45	60	67	64	53	35	20	13	12	11	9	7	5	3	2	1	0	0	0	0	0	0	0	0	0
0	28	52	69	76	72	59	38	21	13	12	12	11	9	7	5	3	2	1	0	0	0	0	0	0	0	0
0	31	58	76	84	79	64	40	21	12	12	12	11	10	8	6	4	3	2	1	0	0	0	0	0	0	0
0	34	63	83	90	85	67	41	20	11	10	11	11	10	9	7	5	4	3	2	1	1	0	0	0	0	1
0	36	67	87	95	89	69	41	18	8	7	9	10	10	9	8	6	5	4	3	2	1	1	1	1	1	1
0	37	69	91	98	90	68	39	15	4	4	6	8	8	8	8	7	6	5	4	3	2	2	1	1	1	1
0	38	71	93	100	90	67	36	11	0	0	2	4	6	7	7	7	7	6	5	4	3	2	2	1	1	1

SIGMA1 4

*** MAXIMUM VALUE =-0.34042E+00 ***

2	2	2	2	2	2	2	1	1	1	1	1	0	0	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	0
2	2	2	2	2	2	2	2	2	2	1	1	1	0	0	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1
2	2	2	2	2	2	2	2	2	2	2	2	1	1	0	0	0	0	0	0	-1	-1	-1	-1	-1	-1	-1
1	1	1	2	2	2	2	2	2	2	2	2	2	1	1	0	0	0	0	0	0	-1	-1	-1	-1	-1	-1
1	1	1	2	2	2	2	3	3	3	2	2	2	2	1	1	1	0	0	0	0	0	-1	-1	-1	-1	-1
1	1	1	1	2	2	2	3	3	2	2	2	2	2	2	1	1	1	0	0	0	0	0	-1	-1	-1	-1
0	0	0	1	1	2	2	2	2	2	2	2	2	2	2	2	1	1	1	0	0	0	0	0	0	-1	-1
0	0	0	0	0	1	2	2	2	2	2	2	2	2	2	2	2	1	1	1	1	0	0	0	0	0	-1
0	0	0	0	0	0	0	1	1	1	2	2	2	2	2	2	2	2	2	1	1	1	0	0	0	0	0
2	2	2	1	0	0	0	0	0	1	1	1	2	2	2	2	2	2	2	2	1	1	1	0	0	0	0
11	11	9	6	3	0	-1	-2	-1	0	0	1	1	1	2	2	2	2	2	2	2	1	1	1	1	0	0
26	25	21	16	9	3	0	-3	-3	-2	0	0	1	1	1	2	2	2	2	2	2	2	1	1	1	1	1
44	42	36	28	19	10	2	-2	-3	-3	-2	0	0	1	1	1	2	2	2	2	2	2	2	2	2	2	2
62	60	53	43	31	19	9	2	-1	-3	-3	-2	-1	0	0	1	1	1	1	1	2	2	2	2	2	2	3
79	76	69	58	45	32	19	10	3	0	-2	-2	-2	-1	0	0	1	1	1	1	1	2	2	2	2	2	3
91	89	82	72	59	46	32	21	12	5	1	-1	-2	-2	-1	0	0	0	1	1	1	2	2	3	3	3	3
98	96	91	82	72	59	46	34	23	15	8	3	0	-1	-1	-1	0	0	0	1	2	2	3	3	3	3	3
100	98	95	88	80	70	59	48	37	27	18	10	4	1	0	-1	-1	0	0	1	2	2	3	3	3	3	3
95	95	93	89	85	78	70	61	51	40	30	20	12	6	2	0	0	0	0	1	2	2	3	3	3	2	2
85	85	85	85	84	81	77	71	62	53	42	32	22	14	8	4	1	0	0	1	2	2	3	2	2	1	1
70	71	73	75	78	79	79	76	71	64	54	44	34	25	17	10	6	3	2	1	1	2	2	1	1	0	0
53	54	57	62	67	72	76	77	75	71	64	56	46	37	27	19	13	8	5	3	2	1	0	0	0	0	0
34	36	40	46	53	61	67	72	74	74	71	65	57	48	39	30	22	15	10	5	2	0	0	-1	-1	-1	-1
18	19	24	30	38	47	55	63	69	72	73	70	66	59	51	42	32	24	16	10	5	1	0	-2	-2	-2	-1
3	5	8	14	22	31	41	50	59	66	71	72	71	67	60	52	42	33	23	15	8	3	0	-1	-2	-2	-2
-9	-8	-5	0	6	15	25	36	46	56	64	69	71	70	66	60	51	41	31	21	13	7	2	0	-1	-2	-1
-24	-23	-20	-15	-8	0	10	21	32	44	54	62	67	70	68	64	57	48	38	29	20	13	7	3	0	0	-1